

Part I

Axiomatic K-theory

Throughout *Part I* we endow $\{0, 1\}$ with the structure of a group by identifying it with \mathbf{Z}_2 and take $i \in \{0, 1\}$.

Chapter 1

The Axiomatic Theory

1.1 E - C^* -algebras

DEFINITION 1.1.1 *In this book we call E - C^* -algebra a C^* -algebra F endowed with a bilinear map (exterior multiplication)*

$$E \times F \longrightarrow F, \quad (\alpha, x) \longmapsto \alpha x$$

such that for all $\alpha, \beta \in E$ and $x, y \in F$,

$$\begin{aligned} (\alpha + \beta)x &= \alpha x + \beta x, & (\alpha\beta)x &= \alpha(\beta x), & (\alpha x)^* &= \alpha^* x^*, & \|\alpha x\| &\leq \|\alpha\| \|x\|, \\ \alpha(x + y) &= \alpha x + \alpha y, & \alpha(xy) &= (\alpha x)y = x(\alpha y), & 1_E x &= x. \end{aligned}$$

An E - C^* -subalgebra (E -ideal) of F is a C^* -subalgebra (a closed ideal) G of F such that

$$(\alpha, x) \in E \times G \implies \alpha x \in G.$$

If F, G are E - C^* -algebras then a C^* -homomorphism $\phi : F \longrightarrow G$ is called **E -linear** or an **E - C^* -homomorphism** if for all $(\alpha, x) \in E \times F$, $\phi(\alpha x) = \alpha \phi x$. A bijective E - C^* -homomorphism is called **E - C^* -isomorphism**. We denote by 0 the E - C^* -algebra having a unique element. We denote by \mathfrak{M}_E the category of E - C^* -algebras for which the morphisms are the E -linear C^* -homomorphisms. In particular $\mathfrak{M}_{\mathbb{C}}$ is the category of all C^* -algebras.

If G is an E -ideal of the E - C^* -algebra F then the C^* -algebra F/G has a natural structure of an E - C^* -algebra and

$$0 \longrightarrow G \xrightarrow{\phi} F \xrightarrow{\psi} F/G \longrightarrow 0$$

is an exact sequence in \mathfrak{M}_E , where ϕ denotes the inclusion map and ψ the quotient map. Conversely, if

$$0 \longrightarrow F \xrightarrow{\phi} G \xrightarrow{\psi} H \longrightarrow 0$$

is an exact sequence in \mathfrak{M}_E then F is an E -ideal of G and $H \approx G/F$.

DEFINITION 1.1.2 If $(F_j)_{j \in J}$ is a finite family of E - C^* -algebras then we denote by $\prod_{j \in J} F_j$ the E - C^* -algebra obtained by endowing the corresponding C^* -algebra $\prod_{j \in J} F_j$ with the bilinear map

$$E \times \prod_{j \in J} F_j \longrightarrow \prod_{j \in J} F_j, \quad (\alpha, (x_j)_{j \in J}) \longmapsto (\alpha x_j)_{j \in J}.$$

PROPOSITION 1.1.3 *Every C^* -algebra can be endowed with the structure of an E - C^* -algebra.*

Let F be a C^* -algebra. Let Ω be the spectrum of E and $\omega \in \Omega$ and put

$$E \times F \longrightarrow F, \quad (\alpha, x) \longmapsto \alpha(\omega)x.$$

It is easy to see that F endowed with this exterior multiplication is an E - C^* -algebra. ■

EXAMPLE 1.1.4 *Let Ω be a finite set and $E := \mathcal{C}(\Omega, \mathbb{C})$.*

a) *Let $(F_\omega)_{\omega \in \Omega}$ be a finite family of C^* -algebras and $F := \prod_{\omega \in \Omega} F_\omega$. If we put for all $(\alpha, x) \in E \times F$,*

$$\alpha x : \Omega \longrightarrow F, \quad \omega \longmapsto \alpha(\omega)x_\omega$$

then F endowed with the exterior multiplication

$$E \times F \longrightarrow F, \quad (\alpha, x) \longmapsto \alpha x$$

is an E - C^ -algebra.*

b) *Let F be an E - C^* -algebra and for every $\omega \in \Omega$ put*

$$e_\omega : \Omega \longrightarrow \mathbb{C}, \quad \omega' \longmapsto \begin{cases} 1 & \text{if } \omega' = \omega \\ 0 & \text{if } \omega' \neq \omega \end{cases},$$

$$F_\omega := \{ e_\omega x \mid x \in F \}.$$

Then F_ω is a C^ -algebra for all $\omega \in \Omega$ and $F \approx \prod_{\omega \in \Omega} F_\omega$, with the meaning of a). ■*

EXAMPLE 1.1.5 *Let Ω be a discrete locally compact space, Ω^* a compactification of Ω , $E := \mathcal{C}(\Omega^*, \mathbb{C})$, $(F_\omega)_{\omega \in \Omega}$ a family of C^* -algebras, and $F := \prod_{\omega \in \Omega} F_\omega$ (resp. $F := \left\{ x \in \prod_{\omega \in \Omega} F_\omega \mid \lim_{\omega \rightarrow \infty} \|x_\omega\| = 0 \right\}$). If we put for all $(\alpha, x) \in E \times F$*

$$\alpha x : \Omega \longrightarrow F, \quad \omega \longmapsto \alpha(\omega)x_\omega$$

then $\alpha x \in F$ for all $(\alpha, x) \in E \times F$ and F endowed with the exterior multiplication

$$E \times F \longrightarrow F, \quad (\alpha, x) \longmapsto \alpha x$$

is an E - C^ -algebra. ■*

1.2 The Axioms

DEFINITION 1.2.1 We denote by K_0 and K_1 two covariant functors from the category \mathfrak{M}_E to the category of additive groups. We denote by 0 the group which has a unique element and call **K-null** an E -C*-algebra F for which $K_i(F) = 0$. Let $F \xrightarrow{\varphi} G$ be a morphism in \mathfrak{M}_E . We say that φ is **K-null** if $K_i(\varphi) = 0$. We say that φ **factorizes through null** if there are morphisms $F \xrightarrow{\varphi'} H$ and $H \xrightarrow{\varphi''} G$ in \mathfrak{M}_E such that $\varphi = \varphi'' \circ \varphi'$ and such that H is K-null.

We have $K_i(id_F) = id_{K_i(F)}$ for every E -C*-algebra F . Every morphism which factorizes through null is K-null.

AXIOM 1.2.2 (Null-axiom) $K_i(0) = 0$.

AXIOM 1.2.3 (Split exact axiom) If

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightleftharpoons[\lambda]{\psi} H \longrightarrow 0$$

is a split exact sequence in \mathfrak{M}_E then

$$0 \longrightarrow K_i(F) \xrightarrow{K_i(\varphi)} K_i(G) \xrightleftharpoons[\leftarrow K_i(\lambda)]{K_i(\psi)} K_i(H) \longrightarrow 0$$

is a split exact sequence in the category of additive groups.

It follows that the map

$$K_i(F) \times K_i(H) \longrightarrow K_i(G), \quad (a, b) \longmapsto K_i(\varphi)a + K_i(\lambda)b$$

is a group isomorphism.

DEFINITION 1.2.4 Let $\varphi, \psi : F \longrightarrow G$ be morphisms in \mathfrak{M}_E . We say that φ and ψ are **homotopic** if there is a path

$$\phi_s : F \longrightarrow G, \quad s \in [0, 1]$$

of morphisms in \mathfrak{M}_E such that $\phi_0 = \varphi$, $\phi_1 = \psi$, and the map

$$[0, 1] \longrightarrow G, \quad s \longmapsto \phi_s x$$

is continuous for every $x \in F$.

We say that a pair $F \xrightarrow{\varphi} G$, $G \xrightarrow{\psi} F$ of morphisms in \mathfrak{M}_E is a **homotopy** if $\psi \circ \varphi$ is homotopic to id_F and $\varphi \circ \psi$ is homotopic to id_G . In this case we say that F and G are **homotopic**. F is called **null-homotopic** if it is homotopic to the E - C^* -algebra 0 .

AXIOM 1.2.5 (Homotopy axiom) If $\varphi, \psi : F \longrightarrow G$ are homotopic morphisms in \mathfrak{M}_E then $K_i(\varphi) = K_i(\psi)$.

DEFINITION 1.2.6 We associate to every exact sequence

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 0$$

in \mathfrak{M}_E two group homomorphisms (called **index maps**)

$$\delta_i : K_i(H) \longrightarrow K_{i+1}(F) .$$

AXIOM 1.2.7 (Six-term axiom) For every exact sequence in \mathfrak{M}_E

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 0$$

the six-term sequence

$$\begin{array}{ccccc} K_0(F) & \xrightarrow{K_0(\varphi)} & K_0(G) & \xrightarrow{K_0(\psi)} & K_0(H) \\ \delta_1 \uparrow & & & & \downarrow \delta_0 \\ K_1(H) & \xleftarrow{K_1(\psi)} & K_1(G) & \xleftarrow{K_1(\varphi)} & K_1(F) \end{array}$$

is exact.

AXIOM 1.2.8 (Commutativity of the index maps) If the diagram in \mathfrak{M}_E

$$\begin{array}{ccccccc} 0 & \longrightarrow & F & \xrightarrow{\varphi} & G & \xrightarrow{\psi} & H \longrightarrow 0 \\ & & \phi_1 \downarrow & & \phi_2 \downarrow & & \downarrow \phi_3 \\ 0 & \longrightarrow & F' & \xrightarrow{\varphi'} & G' & \xrightarrow{\psi'} & H' \longrightarrow 0 \end{array}$$

is commutative and has exact rows then the diagram

$$\begin{array}{ccc} K_i(H) & \xrightarrow{\delta_i} & K_{i+1}(F) \\ K_i(\phi_3) \downarrow & & \downarrow K_{i+1}(\phi_1) \\ K_i(H') & \xrightarrow{\delta'_i} & K_{i+1}(F') \end{array}$$

is commutative, where δ_i and δ'_i denote the index maps associated to the upper and the lower row of the above diagram, respectively.

Remark. The above axioms are fulfilled if $K_i(F) = 0$ for all E - C^* -algebras F .

1.3 Some Elementary Results

PROPOSITION 1.3.1 *If*

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightleftharpoons[\lambda]{\psi} H \longrightarrow 0$$

is a split exact sequence in \mathfrak{M}_E then its index maps are 0.

By the split exact axiom (Axiom 1.2.3),

$$0 \longrightarrow K_i(F) \xrightarrow{K_i(\varphi)} K_i(G) \xrightleftharpoons[\leftarrow K_i(\lambda)]{\rightarrow K_i(\psi)} K_i(H) \longrightarrow 0$$

is a split exact sequence in the category of additive groups and the assertion follows from the six-term axiom (Axiom 1.2.7). ■

DEFINITION 1.3.2 *Let $(F_j)_{j \in J}$ be a finite family of E - C^* -algebras, $F := \prod_{j \in J} F_j$ and for every $j \in J$ let $\varphi_j : F_j \longrightarrow F$ be the canonical inclusion and $\psi_j : F \longrightarrow F_j$ the canonical projection. We define*

$$\begin{aligned} \Phi_{(F_j)_{j \in J}, i} : \prod_{j \in J} K_i(F_j) &\longrightarrow K_i(F), \quad (a_j)_{j \in J} \longmapsto \sum_{j \in J} K_i(\varphi_j) a_j, \\ \Psi_{(F_j)_{j \in J}, i} : K_i(F) &\longrightarrow \prod_{j \in J} K_i(F_j), \quad a \longmapsto (K_i(\psi_j) a)_{j \in J}. \end{aligned}$$

PROPOSITION 1.3.3 *If $(F_j)_{j \in J}$ is a finite family of E - C^* -algebras then the map*

$$\Phi_{(F_j)_{j \in J}, i} : \prod_{j \in J} K_i(F_j) \longrightarrow K_i \left(\prod_{j \in J} F_j \right)$$

is a group isomorphism and

$$\Psi_{(F_j)_{j \in J}, i} : K_i \left(\prod_{j \in J} F_j \right) \longrightarrow \prod_{j \in J} K_i(F_j)$$

is its inverse.

If $J = \emptyset$ then the assertion follows from the null-axiom (Axiom 1.2.2). The assertion is trivial for $\text{Card } J = 1$. We prove the general case by induction with respect to $\text{Card } J$. Let $j_0 \in J$ and assume the assertion holds for $J' := J \setminus \{j_0\}$. We denote by

$$\varphi : F_{j_0} \longrightarrow \prod_{j \in J} F_j, \quad \lambda : \prod_{j \in J'} F_j \longrightarrow \prod_{j \in J} F_j$$

the canonical inclusion maps and by

$$\psi : \prod_{j \in J} F_j \longrightarrow \prod_{j \in J'} F_j$$

the canonical projection. Then

$$0 \longrightarrow F_{j_0} \xrightarrow{\varphi} \prod_{j \in J} F_j \xrightleftharpoons[\lambda]{\psi} \prod_{j \in J'} F_j \longrightarrow 0$$

is a split exact sequence in \mathfrak{M}_E . By the split exact axiom (Axiom 1.2.3) the map

$$\Psi_i : K_i(F_{j_0}) \times K_i \left(\prod_{j \in J'} F_j \right) \longrightarrow K_i \left(\prod_{j \in J} F_j \right), \quad (a, b) \longmapsto K_i(\varphi)a + K_i(\lambda)b$$

is a group isomorphism. Since

$$\Psi_i \circ (id_{K_i(F_{j_0})} \times \Phi_{(F_j)_{j \in J'}, i}) = \Phi_{(F_j)_{j \in J}, i}$$

it follows from the induction hypothesis that $\Phi_{(F_j)_{j \in J}, i}$ is a group isomorphism.

The last assertion follows from $\psi_j \circ \varphi_j = id_{F_j}$ for every $j \in J$ and

$$\sum_{j \in J} \varphi_j \circ \psi_j = id_{\prod_{j \in J} F_j}.$$

■

PROPOSITION 1.3.4 *Let $(F_j \xrightarrow{\phi_j} F'_j)_{j \in J}$ be a finite family of morphisms in \mathfrak{M}_E ,*

$$F := \prod_{j \in J} F_j, \quad F' := \prod_{j \in J} F'_j,$$

and for every $j \in J$ let

$$\varphi_j : F_j \longrightarrow F, \quad \varphi'_j : F'_j \longrightarrow F'$$

be the inclusion maps. Then the diagram

$$\begin{array}{ccc} \prod_{j \in J} K_i(F_j) & \xrightarrow{\sum_{j \in J} K_i(\varphi_j)} & K_i(F) \\ \prod_{j \in J} K_i(\phi_j) \downarrow & & \downarrow K_i\left(\prod_{j \in J} \phi_j\right) \\ \prod_{j \in J} K_i(F'_j) & \xrightarrow{\sum_{j \in J} K_i(\varphi'_j)} & K_i(F') \end{array}$$

is commutative.

For every $j \in J$ the diagram

$$\begin{array}{ccc} F_j & \xrightarrow{\varphi_j} & F \\ \phi_j \downarrow & & \downarrow \prod_{j \in J} \phi_j \\ F'_j & \xrightarrow{\varphi'_j} & F' \end{array}$$

is commutative so the diagram

$$\begin{array}{ccc} K_i(F_j) & \xrightarrow{K_i(\varphi_j)} & K_i(F) \\ K_i(\phi_j) \downarrow & & \downarrow K_i\left(\prod_{j \in J} \phi_j\right) \\ K_i(F'_j) & \xrightarrow{K_i(\varphi'_j)} & K_i(F') \end{array}$$

is also commutative. For $(a_j)_{j \in J} \in \prod_{j \in J} K_i(F_j)$, by the above,

$$K_i\left(\prod_{j \in J} \phi_j\right) \circ \left(\sum_{j \in J} K_i(\varphi_j)\right) (a_j)_{j \in J} = K_i\left(\prod_{j \in J} \phi_j\right) \sum_{j \in J} K_i(\varphi_j) a_j =$$

$$\begin{aligned}
 &= \sum_{j \in J} K_i \left(\prod_{k \in J} \phi_k \right) K_i(\phi_j) a_j = \sum_{j \in J} K_i(\phi'_j) K_i(\phi_j) a_j = \\
 &= \left(\sum_{j \in J} K_i(\phi'_j) \right) (K_i(\phi_j) a_j)_{j \in J} = \left(\sum_{j \in J} K_i(\phi'_j) \right) K_i \left(\prod_{j \in J} \phi_j \right) (a_j)_{j \in J},
 \end{aligned}$$

which proves the assertion. ■

PROPOSITION 1.3.5

a) If $F \xrightarrow{\phi} G, G \xrightarrow{\psi} F$ is a homotopy in \mathfrak{M}_E then

$$K_i(\phi) \circ K_i(\psi) = id_{K_i(G)}, \quad K_i(\psi) \circ K_i(\phi) = id_{K_i(F)}.$$

b) If F and G are homotopic E - C^* -algebras then $K_i(F)$ and $K_i(G)$ are isomorphic.

c) If the E - C^* -algebra F is null-homotopic then it is K -null.

a) follows from the homotopy axiom (Axiom 1.2.5).

b) follows from a).

c) follows from b) and from the null-axiom (Axiom 1.2.2). ■

PROPOSITION 1.3.6 Let

$$0 \longrightarrow F \xrightarrow{\phi} G \xrightarrow{\psi} H \longrightarrow 0$$

be an exact sequence in \mathfrak{M}_E .

a) If F (resp. H) is K -null then

$$K_i(G) \xrightarrow{K_i(\psi)} K_i(H) \quad (\text{resp. } K_i(F) \xrightarrow{K_i(\phi)} K_i(G))$$

is a group isomorphism.

b) If G is K -null then

$$K_i(H) \xrightarrow{\delta_i} K_{i+1}(F)$$

is a group isomorphism.

c) If φ is K -null then the sequences

$$0 \longrightarrow K_i(G) \xrightarrow{K_i(\psi)} K_i(H) \xrightarrow{\delta_i} K_{i+1}(F) \longrightarrow 0$$

is exact.

d) If ψ is K -null then the sequences

$$0 \longrightarrow K_i(H) \xrightarrow{\delta_i} K_{i+1}(F) \xrightarrow{K_{i+1}(\varphi)} K_{i+1}(G) \longrightarrow 0$$

is exact.

e) The index maps of a split exact sequence are equal to 0.

a), b), c), and d) follow from the six-term axiom (Axiom 1.2.7).

e) follows from the six-term axiom (Axiom 1.2.7) and from the split exact axiom (Axiom 1.2.3). ■

PROPOSITION 1.3.7 An \mathfrak{M}_E -triple is a triple (F_1, F_2, F_3) such that F_1 is an E - C^* -algebra, F_2 is an E -ideal of F_1 , and F_3 is an E -ideal of F_1 and of F_2 . We denote for all $j, k \in \mathbb{N}_3$, $j < k$, by $\varphi_{j,k} : F_k \longrightarrow F_j$ the inclusion map, by $\psi_{j,k} : F_j \longrightarrow F_j/F_k$ the quotient map, and by $\delta_{j,k,i} : K_i(F_j/F_k) \longrightarrow F_k$ the index maps associated to the exact sequence in \mathfrak{M}_E

$$0 \longrightarrow F_k \xrightarrow{\varphi_{j,k}} F_j \xrightarrow{\psi_{j,k}} F_j/F_k \longrightarrow 0.$$

a) There is a unique morphism $F_2/F_3 \xrightarrow{\varphi_{1,2}/F_3} F_1/F_3$ in \mathfrak{M}_E such that

$$\psi_{1,3} \circ \varphi_{1,2} = (\varphi_{1,2}/F_3) \circ \psi_{2,3}.$$

b) The diagram

$$\begin{array}{ccccccc} K_i(F_3) & \xrightarrow{K_i(\varphi_{1,3})} & K_i(F_1) & \xrightarrow{K_i(\psi_{1,3})} & K_i(F_1/F_3) & \xrightarrow{\delta_{1,3,i}} & K_{i+1}(F_3) \\ = \uparrow & & \uparrow K_i(\varphi_{1,2}) & & \uparrow K_i(\varphi_{1,2}/F_3) & & \uparrow = \\ K_i(F_3) & \xrightarrow{K_i(\varphi_{2,3})} & K_i(F_2) & \xrightarrow{K_i(\psi_{2,3})} & K_i(F_2/F_3) & \xrightarrow{\delta_{2,3,i}} & K_{i+1}(F_3) \end{array}$$

is commutative.

a) is easy to see.

b) follows from a), $\varphi_{1,2} \circ \varphi_{2,3} = \varphi_{1,3}$, and from the axiom of commutativity of the index maps (Axiom 1.2.8). ■

THEOREM 1.3.8 (The triple theorem) *Let (F_1, F_2, F_3) be an \mathfrak{M}_E -triple.*

a) *Assume F_2 K -null.*

$a_1)$ $\delta_{2,3,i} : K_i(F_2/F_3) \longrightarrow K_{i+1}(F_3)$ *is a group isomorphism.*

$a_2)$ $\delta_{2,3,i} = \delta_{1,3,i} \circ K_i(\varphi_{1,2}/F_3)$.

$a_3)$ $\varphi_{1,3}$ *is K -null.*

$a_4)$ *If we put $\Phi_i := K_i(\varphi_{1,2}/F_3) \circ (\delta_{2,3,i})^{-1}$ then*

$$0 \longrightarrow K_i(F_1) \xrightarrow{K_i(\psi_{1,3})} K_i(F_1/F_3) \xrightarrow[\leftarrow]{\delta_{1,3,i}} K_{i+1}(F_3) \longrightarrow 0$$

is a split exact sequence and the map

$$K_i(F_1) \times K_{i+1}(F_3) \longrightarrow K_i(F_1/F_3), \quad (a, b) \longmapsto K_i(\psi_{1,3})a + \Phi_i b$$

is a group isomorphism.

b) *Assume F_1/F_3 K -null.*

$b_1)$ $\delta_{2,3,i} = 0$ *and the sequence*

$$0 \longrightarrow K_i(F_3) \xrightarrow{K_i(\varphi_{2,3})} K_i(F_2) \xrightarrow{K_i(\psi_{2,3})} K_i(F_2/F_3) \longrightarrow 0$$

is exact.

$b_2)$ $K_i(\varphi_{1,3}) : K_i(F_3) \longrightarrow K_i(F_1)$ *is a group isomorphism.*

$b_3)$ *If we put $\Phi_i := K_i(\varphi_{1,3})^{-1} \circ K_i(\varphi_{1,2})$ then the map*

$$\Psi : K_i(F_2) \longrightarrow K_i(F_3) \times K_i(F_2/F_3), \quad b \longmapsto (\Phi_i b, K_i(\psi_{2,3})b)$$

is a group isomorphism.

$b_4)$ If $\psi_{1,2}$ is K -null and if we put $\Phi'_i := K_i(\varphi_{2,3}) \circ K_i(\varphi_{1,3})^{-1}$ then

$$0 \longrightarrow K_{i+1}(F_1/F_2) \xrightarrow{\delta_{1,2,(i+1)}} K_i(F_2) \xrightleftharpoons[\Phi'_i]{K_i(\varphi_{1,2})} K_i(F_1) \longrightarrow 0$$

is a split exact sequence and the map

$$K_i(F_1) \times K_{i+1}(F_1/F_2) \longrightarrow K_i(F_2), \quad (a, b) \longmapsto \Phi'_i a + \delta_{1,2,(i+1)} b$$

is a group isomorphism.

$c)$ Assume F_1 K -null and denote by ψ the canonical map $F_1/F_3 \rightarrow F_1/F_2$.

$c_1)$ $\delta_{1,2,i}$ and $\delta_{1,3,i}$ are group isomorphisms.

$c_2)$ $K_i(\varphi_{2,3}) \circ \delta_{1,3,(i+1)} = \delta_{1,2,(i+1)} \circ K_{i+1}(\psi)$.

$c_3)$ Let $\varphi : F_1/F_2 \longrightarrow F_1/F_3$ be a morphism in \mathfrak{M}_E such that

$$K_i(\psi \circ \varphi) = id_{K_i(F_1/F_2)}.$$

If we put

$$\Phi_i := \delta_{1,3,(i+1)} \circ K_{i+1}(\varphi) \circ (\delta_{1,2,(i+1)})^{-1}$$

then $K_i(\varphi_{2,3}) \circ \Phi_i = id_{K_i(F_2)}$. If in addition $\psi_{2,3}$ is K -null then

$$0 \longrightarrow K_{i+1}(F_2/F_3) \xrightarrow{\delta_{2,3,(i+1)}} K_i(F_3) \xrightleftharpoons[\Phi_i]{K_i(\varphi_{2,3})} K_i(F_2) \longrightarrow 0$$

is a split exact sequence and the map

$$K_{i+1}(F_2/F_3) \times K_i(F_2) \longrightarrow K_i(F_3), \quad (a, b) \longmapsto \delta_{2,3,(i+1)} a + \Phi_i b$$

is a group isomorphism.

$a_1)$ follows from Proposition 1.3.6 b).

$a_2)$ follows from Proposition 1.3.7 b).

$a_3)$ $\varphi_{1,3}$ factorizes through null and so it is K -null.

$a_4)$ By $a_2)$,

$$\delta_{1,3,i} \circ \Phi_i = \delta_{1,3,i} \circ K_i(\varphi_{1,2}/F_3) \circ (\delta_{2,3,i})^{-1} = \delta_{2,3,i} \circ (\delta_{2,3,i})^{-1} = id_{K_i(F_3)}$$

and this implies the assertion.

$b_1)$ By Proposition 1.3.7 b), $\delta_{2,3,i}$ factorizes through null and so it is K-null. By the six-term axiom (Axiom 1.2.7) the sequence

$$0 \longrightarrow K_i(F_3) \xrightarrow{K_i(\varphi_{2,3})} K_i(F_2) \xrightarrow{K_i(\psi_{2,3})} K_i(F_2/F_3) \longrightarrow 0$$

is exact.

$b_2)$ follows from Proposition 1.3.6 a).

$b_3)$ Step 1

$$\Phi_i \circ K_i(\varphi_{2,3}) = id_{K_i(F_3)}$$

Since $\varphi_{1,3} = \varphi_{1,2} \circ \varphi_{2,3}$,

$$\begin{aligned} \Phi_i \circ K_i(\varphi_{2,3}) &= K_i(\varphi_{1,3})^{-1} \circ K_i(\varphi_{1,2}) \circ K_i(\varphi_{2,3}) = \\ &= K_i(\varphi_{1,3})^{-1} \circ K_i(\varphi_{1,3}) = id_{K_i(F_3)} . \end{aligned}$$

Step 2 Ψ is injective

Let $b \in K_i(F_2)$ with $\Psi b = 0$. Then $K_i(\psi_{2,3})b = 0$ so by $b_1)$,

$$b \in \text{Ker } K_i(\psi_{2,3}) = \text{Im } K_i(\varphi_{2,3})$$

and there is an $a \in K_i(F_3)$ with $b = K_i(\varphi_{2,3})a$. By Step 1,

$$a = \Phi_i K_i(\varphi_{2,3})a = \Phi_i b = 0,$$

so $b = 0$ and Ψ is injective.

Step 3 Ψ is surjective

Let $(a, c) \in K_i(F_3) \times K_i(F_2/F_3)$. Put $b' := K_i(\varphi_{2,3})a$. By $b_1)$,

$$K_i(\psi_{2,3})b' = K_i(\psi_{2,3})K_i(\varphi_{2,3})a = 0$$

and by Step 1, $\Phi_i b' = \Phi_i K_i(\varphi_{2,3})a = a$. By b_1), there is a $b'' \in K_i(F_2)$ with $c = K_i(\psi_{2,3})b''$. By Step 1,

$$\Phi_i(b'' - K_i(\varphi_{2,3})\Phi_i b'') = \Phi_i b'' - \Phi_i K_i(\varphi_{2,3})\Phi_i b'' = \Phi_i b'' - \Phi_i b'' = 0.$$

Thus by b_1),

$$\begin{aligned} \Psi(b' + b'' - K_i(\varphi_{2,3})\Phi_i b'') &= \\ &= (\Phi_i b', K_i(\psi_{2,3})b'' - K_i(\psi_{2,3})K_i(\varphi_{2,3})\Phi_i b'') = (a, c) \end{aligned}$$

and Ψ is surjective.

b_4) Since $\varphi_{1,3} = \varphi_{1,2} \circ \varphi_{2,3}$,

$$\begin{aligned} K_i(\varphi_{1,2}) \circ \Phi_i' &= K_i(\varphi_{1,2}) \circ K_i(\varphi_{2,3}) \circ K_i(\varphi_{1,3})^{-1} = \\ &= K_i(\varphi_{1,3}) \circ K_i(\varphi_{1,3})^{-1} = id_{K_i(F_1)} \end{aligned}$$

and the assertion follows.

c_1) follows from Proposition 1.3.6 b)).

c_2) follows from the commutativity of the index maps (Axiom 1.2.8).

c_3) By c_2),

$$\begin{aligned} K_i(\varphi_{2,3}) \circ \Phi_i &= K_i(\varphi_{2,3}) \circ \delta_{1,3,(i+1)} \circ K_{i+1}(\varphi) \circ (\delta_{1,2,(i+1)})^{-1} = \\ &= \delta_{1,2,(i+1)} \circ K_{i+1}(\psi) \circ K_{i+1}(\varphi) \circ (\delta_{1,2,(i+1)})^{-1} = \\ &= \delta_{1,2,(i+1)} \circ K_{i+1}(\psi \circ \varphi) \circ (\delta_{1,2,(i+1)})^{-1} = \delta_{1,2,(i+1)} \circ (\delta_{1,2,(i+1)})^{-1} = id_{K_i(F_2)}. \end{aligned}$$

The last assertion follows from the first one. ■

Remark. a) still holds with the weaker assumption that F_2 is only an E - C^* -subalgebra of F_1 .

1.4 Tensor Products

Throughout this section F denotes an E - C^* -algebra.

DEFINITION 1.4.1 Let G be a C^* -algebra. We denote by $F \otimes G$ the spatial tensor product of F and G endowed with the structure of an E - C^* -algebra by using the exterior multiplication

$$E \times (F \otimes G) \longrightarrow F \otimes G, \quad (\alpha, x \otimes y) \longmapsto (\alpha x) \otimes y$$

([5] Proposition T.5.14 and T.5.17 Remark). If $F \xrightarrow{\varphi} F'$ is a morphism in \mathfrak{M}_E and $G \xrightarrow{\psi} G'$ a morphism in $\mathfrak{M}_{\mathbb{C}}$ then $F \otimes G \xrightarrow{\varphi \otimes \psi} F' \otimes G'$ denotes the morphism in \mathfrak{M}_E defined by

$$\varphi \otimes \psi : F \otimes G \longrightarrow F' \otimes G', \quad x \otimes y \longmapsto \varphi x \otimes \psi y.$$

If $(G_j)_{j \in J}$ is a family of C^* -algebras then we put

$$\bigotimes_{j \in \emptyset} G_j := \mathbb{C}.$$

We have $F \otimes \mathbb{C} \approx F$ and $id_F \otimes id_G = id_{F \otimes G}$. If $F \xrightarrow{\varphi} F' \xrightarrow{\varphi'} F''$ are morphisms in \mathfrak{M}_E and $G \xrightarrow{\psi} G' \xrightarrow{\psi'} G''$ are morphisms in $\mathfrak{M}_{\mathbb{C}}$ then

$$(\varphi \otimes \psi) \circ (\varphi' \otimes \psi') = (\varphi \circ \varphi') \otimes (\psi \circ \psi').$$

If G and H are C^* -algebras then

$$F \otimes (G \times H) \approx (F \otimes G) \times (F \otimes H), \quad F \otimes (G \otimes H) \approx (F \otimes G) \otimes H.$$

If G is a C^* -algebra and F_1, F_2 are E - C^* -algebras then

$$(F_1 \times F_2) \otimes G \approx (F_1 \otimes G) \times (F_2 \otimes G).$$

PROPOSITION 1.4.2 Let G, H be C^* -algebras.

a) If $\varphi_0, \varphi_1 : G \longrightarrow H$ are homotopic C^* -homomorphisms then $id_F \otimes \varphi_0$ and $id_F \otimes \varphi_1$ are also homotopic.

b) If $G \xrightarrow{\varphi} H, H \xrightarrow{\psi} G$ is a homotopy in $\mathfrak{M}_{\mathbb{C}}$ then

$$F \otimes G \xrightarrow{id_F \otimes \varphi} F \otimes H, \quad F \otimes H \xrightarrow{id_F \otimes \psi} F \otimes G$$

is a homotopy in \mathfrak{M}_E .

c) If G is homotopic to 0 then $F \otimes G$ is also homotopic to 0 and so K -null.

a) Let $[0, 1] \longrightarrow \varphi_s$ be a pointwise continuous map of C^* -homomorphisms $G \rightarrow H$. Let $z \in F \odot G$. There are finite families $(x_j)_{j \in J}$ in F and $(y_j)_{j \in J}$ in G such that

$$z = \sum_{j \in J} x_j \otimes y_j.$$

For $s \in [0, 1]$,

$$(id_F \otimes \varphi_s)z = \sum_{j \in J} x_j \otimes \varphi_s y_j$$

so the map

$$[0, 1] \longrightarrow F \otimes H, \quad s \longmapsto (id_F \otimes \varphi_s)z$$

is continuous.

Let now $z \in F \otimes G$, $s_0 \in [0, 1]$, and $\varepsilon > 0$. There is a $z' \in F \odot G$ such that $\|z - z'\| < \frac{\varepsilon}{3}$. By the above, there is a $\delta > 0$ such that

$$\|(id_F \otimes \varphi_s)z' - (id_F \otimes \varphi_{s_0})z'\| < \frac{\varepsilon}{3}$$

for all $s \in [0, 1]$, $|s - s_0| < \delta$. It follows

$$\begin{aligned} \|(id_F \otimes \varphi_s)z - (id_F \otimes \varphi_{s_0})z\| &\leq \|(id_F \otimes \varphi_s)(z - z')\| + \\ &+ \|(id_F \otimes \varphi_s)z' - (id_F \otimes \varphi_{s_0})z'\| + \|(id_F \otimes \varphi_{s_0})(z - z')\| < \varepsilon, \end{aligned}$$

which proves the assertion.

b) follows from a).

c) follows from b) and Proposition 1.3.5 c)). ■

PROPOSITION 1.4.3 *Let*

$$0 \longrightarrow G_1 \xrightarrow{\varphi} G_2 \xrightleftharpoons[\lambda]{\psi} G_3 \longrightarrow 0$$

be a split exact sequence in $\mathfrak{M}_{\mathbb{C}}$.

a) *The sequence in \mathfrak{M}_E*

$$0 \longrightarrow F \otimes G_1 \xrightarrow{id_F \otimes \varphi} F \otimes G_2 \xrightleftharpoons[id_F \otimes \lambda]{id_F \otimes \psi} F \otimes G_3 \longrightarrow 0$$

is split exact.

b) *The sequence*

$$0 \longrightarrow K_i(F \otimes G_1) \xrightarrow{K_i(id_F \otimes \varphi)} K_i(F \otimes G_2) \xrightleftharpoons[K_i(id_F \otimes \lambda)]{K_i(id_F \otimes \psi)} K_i(F \otimes G_3) \longrightarrow 0$$

is split exact and the map

$$K_i(F \otimes G_1) \times K_i(F \otimes G_3) \longrightarrow K_i(F \otimes G_2),$$

$$(a, b) \longmapsto K_i(id_F \otimes \varphi)a + K_i(id_F \otimes \lambda)b$$

is a group isomorphism.

a) By [5] Corollary T.5.19, $id_F \otimes \varphi$ is injective. We have

$$(id_F \otimes \psi) \circ (id_F \otimes \lambda) = id_F \otimes (\psi \circ \lambda) = id_F \otimes id_{G_3} = id_{F \otimes G_3},$$

$$(id_F \otimes \psi) \circ (id_F \otimes \varphi) = id_F \otimes (\psi \circ \varphi) = 0,$$

so

$$Im(id_F \otimes \varphi) \subset Ker(id_F \otimes \psi).$$

Let $z \in (F \odot G_2) \cap Ker(id_F \otimes \psi)$. There is a linearly independent finite family $(x_j)_{j \in J}$ in F and a family $(y_j)_{j \in J}$ in G_2 such that

$$z = \sum_{j \in J} x_j \otimes y_j.$$

From

$$0 = (id_F \otimes \psi)z = \sum_{j \in J} x_j \otimes \psi y_j$$

we get $\psi y_j = 0$ for all $j \in J$. Thus for every $j \in J$ there is a $y'_j \in G_1$ with $\varphi y'_j = y_j$. It follows

$$z = \sum_{j \in J} x_j \otimes \varphi y'_j = (id_F \otimes \varphi) \sum_{j \in J} x_j \otimes y'_j \in Im(id_F \otimes \varphi).$$

Let $z \in Ker(id_F \otimes \psi)$. Then

$$(id_F \otimes (\lambda \circ \psi))z = (id_F \otimes \lambda)(id_F \otimes \psi)z = 0.$$

Let $(z_n)_{n \in \mathbb{N}}$ be a sequence in $F \odot G_2$ converging to z . For $n \in \mathbb{N}$, by the above,

$$(id_F \otimes \psi)(z_n - (id_F \otimes (\lambda \circ \psi))z_n) =$$

$$\begin{aligned}
&= (id_F \otimes \psi)z_n - (id_F \otimes \psi)(id_F \otimes \lambda)(id_F \otimes \psi)z_n = \\
&= (id_F \otimes \psi)z_n - (id_F \otimes \psi)z_n = 0, \\
&z_n - (id_F \otimes (\lambda \circ \psi))z_n \in Im(id_F \otimes \psi).
\end{aligned}$$

Since $Im(id_F \otimes \phi)$ is closed,

$$z = z - (id_F \otimes (\lambda \circ \psi))z = \lim_{n \rightarrow \infty} (z_n - (id_F \otimes (\lambda \circ \psi))z_n) \in Im(id_F \otimes \phi),$$

which proves the Proposition.

b) follows from a) and the split exact axiom (Axiom 1.2.3). ■

DEFINITION 1.4.4

We denote for every C^* -algebra G by \tilde{G} its unitization (see e.g. [4] Exercise 1.3) and by

$$0 \longrightarrow G \xrightarrow{\iota_G} \tilde{G} \xrightleftharpoons[\lambda_G]{\pi_G} \mathbf{C} \longrightarrow 0$$

its associated split exact sequence. If G and H are C^* -algebras and $\phi : G \longrightarrow H$ is a C^* -homomorphism then $\tilde{\phi} : \tilde{G} \longrightarrow \tilde{H}$ denotes the unitization of ϕ .

COROLLARY 1.4.5 *Let G be a C^* -algebra.*

a) *The sequence in \mathfrak{M}_E*

$$0 \longrightarrow F \otimes G \xrightarrow{id_F \otimes \iota_G} F \otimes \tilde{G} \xrightleftharpoons[id_F \otimes \lambda_G]{id_F \otimes \pi_G} F \longrightarrow 0$$

is split exact.

b) *The sequence*

$$0 \longrightarrow K_i(F \otimes G) \xrightarrow{K_i(id_F \otimes \iota_G)} K_i(F \otimes \tilde{G}) \xrightleftharpoons[K_i(id_F \otimes \lambda_G)]{K_i(id_F \otimes \pi_G)} K_i(F) \longrightarrow 0$$

is split exact and the map

$$K_i(F) \times K_i(F \otimes G) \longrightarrow K_i(F \otimes \tilde{G}),$$

$$(a, b) \longmapsto K_i(id_F \otimes \lambda_G)a + K_i(id_F \otimes \iota_G)b$$

is a group isomorphism.

c) Let $F \xrightarrow{\varphi} F'$ be a morphism in \mathfrak{M}_E and $G \xrightarrow{\psi} G'$ a morphism in $\mathfrak{M}_{\mathbf{C}}$. If we identify the isomorphic groups of b) then

$$K_i(\varphi \otimes \tilde{\psi}) : K_i(F \otimes \tilde{G}) \longrightarrow K_i(F' \otimes \tilde{G}'),$$

$$(a, b) \longmapsto (K_i(\varphi)a, K_i(\varphi \otimes \psi)b)$$

is a group isomorphism.

d) Let $\varphi : G \longrightarrow G'$ be a morphism in $\mathfrak{M}_{\mathbf{C}}$. If we denote by Ψ_i and Ψ'_i the group isomorphisms of b) associated to G and G' , respectively, then

$$K_i(id_F \otimes \tilde{\varphi}) \circ \Psi_i = \Psi'_i \circ (id_{K_i(F)} \times K_i(id_F \otimes \varphi)) .$$

a) and b) follow from Proposition 1.4.3 a), b).

c) follows from b) and the commutativity of the following diagram:

$$\begin{array}{ccccc} F \otimes G & \xrightarrow{id_F \otimes \iota_G} & F \otimes \tilde{G} & \xleftarrow{id_F \otimes \lambda_G} & F \otimes \mathbf{C} \\ \varphi \otimes \psi \downarrow & & \varphi \otimes \tilde{\psi} \downarrow & & \downarrow \varphi \otimes id_{\mathbf{C}} \\ F' \otimes G' & \xrightarrow{id_{F'} \otimes \iota_{G'}} & F' \otimes \tilde{G}' & \xleftarrow{id_{F'} \otimes \lambda_{G'}} & F' \otimes \mathbf{C} \end{array} .$$

d) For $(a, b) \in K_i(F) \times K_i(F \otimes G)$, since $\tilde{\varphi} \circ \lambda_G = \lambda_{G'}$ and $\iota_{G'} \circ \varphi = \tilde{\varphi} \circ \iota_G$,

$$\begin{aligned} K_i(id_F \otimes \tilde{\varphi})\Psi_i(a, b) &= K_i(id_F \otimes \tilde{\varphi})(K_i(id_F \otimes \lambda_G)a + K_i(id_F \otimes \iota_G)b) = \\ &= K_i(id_F \otimes \tilde{\varphi})K_i(id_F \otimes \lambda_G)a + K_i(id_F \otimes \tilde{\varphi})K_i(id_F \otimes \iota_G)b = \\ &= K_i(id_F \otimes (\tilde{\varphi} \circ \lambda_G))a + K_i(id_F \otimes (\tilde{\varphi} \circ \iota_G))b = \\ &= K_i(id_F \otimes \lambda_{G'})a + K_i(id_F \otimes (\iota_{G'} \circ \varphi))b = \\ &= K_i(id_F \otimes \lambda_{G'})a + K_i(id_F \otimes \iota_{G'})K_i(id_F \otimes \varphi)b = \\ &= \Psi'_i(a, K_i(id_F \otimes \varphi)b) = \Psi'_i(id_{K_i(F)} \times K_i(id_F \otimes \varphi))(a, b), \end{aligned}$$

so

$$K_i(id_F \otimes \tilde{\varphi}) \circ \Psi_i = \Psi'_i \circ (id_{K_i(F)} \times K_i(id_F \otimes \varphi)) .$$



PROPOSITION 1.4.6 *If $(G_j)_{j \in J}$ is a finite family of C^* -algebras then*

$$K_i \left(F \otimes \left(\bigotimes_{j \in J} \tilde{G}_j \right) \right) \approx \prod_{I \subset J} K_i \left(F \otimes \left(\bigotimes_{j \in I} G_j \right) \right).$$

We prove the assertion by induction with respect to $\text{Card } J$. The assertion is trivial for $\text{Card } J = 0$ (Definition 1.4.1 and Null-axiom (Axiom 1.4.6)). Let $j_0 \in J$, $J' := J \setminus \{j_0\}$, and assume the assertion holds for J' . By Corollary 1.4.5 b),

$$\begin{aligned} K_i \left(F \otimes \left(\bigotimes_{j \in J} \tilde{G}_j \right) \right) &\approx K_i \left(\left(F \otimes \left(\bigotimes_{j \in J'} \tilde{G}_j \right) \right) \otimes \tilde{G}_{j_0} \right) \approx \\ &\approx K_i \left(F \otimes \left(\bigotimes_{j \in J'} \tilde{G}_j \right) \right) \times K_i \left(\left(F \otimes \left(\bigotimes_{j \in J'} \tilde{G}_j \right) \right) \otimes G_{j_0} \right) \approx \\ &\approx K_i \left(F \otimes \left(\bigotimes_{j \in J'} \tilde{G}_j \right) \right) \times K_i \left((F \otimes G_{j_0}) \otimes \left(\bigotimes_{j \in J'} \tilde{G}_j \right) \right) \approx \\ &\approx \prod_{I \subset J'} K_i \left(F \otimes \left(\bigotimes_{j \in I} G_j \right) \right) \times \prod_{I \subset J'} K_i \left(F \otimes \left(\bigotimes_{j \in I \cup \{j_0\}} G_j \right) \right) \approx \\ &\approx \prod_{I \subset J} K_i \left(F \otimes \left(\bigotimes_{j \in I} G_j \right) \right). \end{aligned}$$

■

COROLLARY 1.4.7 *If G is a C^* -algebra then for all $n \in \mathbb{N}^*$*

$$K_i \left(F \otimes \left(\bigotimes_{j \in \mathbb{N}_n} \tilde{G} \right) \right) \approx \prod_{k=0}^n K_i \left(F \otimes \left(\bigotimes_{j \in \mathbb{N}_k} G \right) \right)^{\binom{n}{k}}.$$

■

PROPOSITION 1.4.8 *Let G be a C^* -algebra and*

$$0 \longrightarrow F_1 \xrightarrow{\varphi} F_2 \xrightarrow[\lambda]{\psi} F_3 \longrightarrow 0$$

a split exact sequence in \mathfrak{M}_E .

a) The sequence in \mathfrak{M}_E

$$0 \longrightarrow F_1 \otimes G \xrightarrow{\varphi \otimes id_G} F_2 \otimes G \xrightleftharpoons[\lambda \otimes id_G]{\psi \otimes id_G} F_3 \otimes G \longrightarrow 0$$

is split exact.

b) The sequence

$$0 \longrightarrow K_i(F_1 \otimes G) \xrightarrow{K_i(\varphi \otimes id_G)} K_i(F_2 \otimes G) \xrightleftharpoons[K_i(\lambda \otimes id_G)]{K_i(\psi \otimes id_G)} K_i(F_3 \otimes G) \longrightarrow 0$$

is split exact and the map

$$K_i(F_1 \otimes G) \times K_i(F_3 \otimes G) \longrightarrow K_i(F_2 \otimes G),$$

$$(a, b) \longmapsto K_i(\varphi \otimes id_G)a + K_i(\lambda \otimes id_G)b$$

is a group isomorphism.

The proof is similar to the proof of Proposition 1.4.3. ■

PROPOSITION 1.4.9 Let

$$0 \longrightarrow G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\psi} G_3 \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_{\mathbb{C}}$. If F or G_3 is nuclear then the sequence in \mathfrak{M}_E

$$0 \longrightarrow F \otimes G_1 \xrightarrow{id_F \otimes \varphi} F \otimes G_2 \xrightarrow{id_F \otimes \psi} F \otimes G_3 \longrightarrow 0$$

is exact and so

$$\frac{F \otimes G_2}{F \otimes G_1} \approx F \otimes \frac{G_2}{G_1}.$$

[5] Theorem T.6.26. ■

PROPOSITION 1.4.10 Let G be a C^* -algebra and

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

an exact sequence in \mathfrak{M}_E . If F_3 or G is nuclear then

$$0 \longrightarrow F_1 \otimes G \xrightarrow{\phi_1 \otimes id_G} F_2 \otimes G \xrightarrow{\phi_2 \otimes id_G} F_3 \otimes G \longrightarrow 0$$

is exact.

[5] Theorem T.6.26. ■

DEFINITION 1.4.11 *Let*

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

be an exact sequence in \mathfrak{M}_E and G a C^ -algebra. If δ_i denotes the index maps associated to the above exact sequence in \mathfrak{M}_E and if the sequence in \mathfrak{M}_E*

$$0 \longrightarrow F_1 \otimes G \xrightarrow{\phi_1 \otimes id_G} F_2 \otimes G \xrightarrow{\phi_2 \otimes id_G} F_3 \otimes G \longrightarrow 0$$

is exact (e.g. F_3 or G is nuclear ([5] T.6.26)) then we denote by $\delta_{G,i}$ the index maps associated to this last exact sequence in \mathfrak{M}_E .

In this case the six-term sequence

$$\begin{array}{ccccc} K_0(F_1 \otimes G) & \xrightarrow{K_0(\phi_1 \otimes id_G)} & K_0(F_2 \otimes G) & \xrightarrow{K_0(\phi_2 \otimes id_G)} & K_0(F_3 \otimes G) \\ \delta_{G,1} \uparrow & & & & \downarrow \delta_{G,0} \\ K_1(F_3 \otimes G) & \xleftarrow{K_1(\phi_2 \otimes id_G)} & K_1(F_2 \otimes G) & \xleftarrow{K_1(\phi_1 \otimes id_G)} & K_1(F_1 \otimes G) \end{array}$$

is exact (by the six-term axiom (Axiom 1.2.7)).

COROLLARY 1.4.12 *Let G be a unital C^* -algebra,*

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

an exact sequence in \mathfrak{M}_E , and δ_i its index maps. We assume that F_3 or G is nuclear and put for every $j \in \{1, 2, 3\}$

$$\varphi_j : F_j \longrightarrow F_j \otimes G, \quad x \longmapsto x \otimes 1_G.$$

Then $\delta_{G,i} \circ K_i(\varphi_3) = K_{i+1}(\varphi_1) \circ \delta_i$.

The diagram

$$\begin{array}{ccccc} F_1 & \xrightarrow{\phi_1} & F_2 & \xrightarrow{\phi_2} & F_3 \\ \varphi_1 \downarrow & & \varphi_2 \downarrow & & \downarrow \varphi_3 \\ F_1 \otimes G & \xrightarrow{\phi_1 \otimes id_G} & F_2 \otimes G & \xrightarrow{\phi_2 \otimes id_G} & F_3 \otimes G \end{array}$$

is commutative and the assertion follows from Proposition 1.4.10 and the commutativity of the index maps (Axiom 1.2.8). ■

1.5 The Class Υ

Throughout this section F denotes an E - C^* -algebra.

DEFINITION 1.5.1 Let Υ be the class of those C^* -algebras G for which there are $p(G), q(G) \in \mathbb{N}^*$ and group isomorphisms

$$\Phi_{i,G,F} : K_i(F)^{p(G)} \times K_{i+1}(F)^{q(G)} \longrightarrow K_i(F \otimes G)$$

such that for every morphism $F \xrightarrow{\phi} F'$ in \mathfrak{M}_E the diagram

$$\begin{array}{ccc} K_i(F)^{p(G)} \times K_{i+1}(F)^{q(G)} & \xrightarrow{\Phi_{i,G,F}} & K_i(F \otimes G) \\ \downarrow K_i(\phi)^{p(G)} \times K_{i+1}(\phi)^{q(G)} & & \downarrow K_i(\phi \otimes id_G) \\ K_i(F')^{p(G)} \times K_{i+1}(F')^{q(G)} & \xrightarrow{\Phi_{i,G,F'}} & K_i(F' \otimes G) \end{array}$$

is commutative. We denote by \vec{G} the class of group isomorphisms

$$\Phi_{i,G,F} : K_i(F)^{p(G)} \times K_{i+1}(F)^{q(G)} \longrightarrow K_i(F \otimes G)$$

having the above property. A C^* -algebra G is called **Υ -null** if $G \in \Upsilon$ and $p(G) = q(G) = 0$.

If G is Υ -null or if F is K -null and $G \in \Upsilon$ then $F \otimes G$ is K -null. In general we shall use $\Phi_{i,G,F}$ without writing $\{\Phi_{i,G,F}\} \in \vec{G}$.

PROPOSITION 1.5.2 Let $p, q \in \mathbb{N}^*$ and let Λ be the class of group isomorphisms

$$\Lambda_{i,F} : K_i(F)^p \times K_{i+1}(F)^q \longrightarrow K_i(F)^p \times K_{i+1}(F)^q$$

such that for all morphisms $F \xrightarrow{\phi} F'$ in \mathfrak{M}_E the diagram

$$\begin{array}{ccc} K_i(F)^p \times K_{i+1}(F)^q & \xrightarrow{\Lambda_{i,F}} & K_i(F)^p \times K_{i+1}(F)^q \\ \downarrow K_i(\phi)^p \times K_{i+1}(\phi)^q & & \downarrow K_i(\phi)^p \times K_{i+1}(\phi)^q \\ K_i(F')^p \times K_{i+1}(F')^q & \xrightarrow{\Lambda_{i,F'}} & K_i(F')^p \times K_{i+1}(F')^q \end{array}$$

is commutative. Let $G \in \Upsilon$ with $p(G) = p$, $q(G) = q$, and let $\{\Phi_{i,G,F}\} \in \vec{G}$.

a) If $\Lambda_{i,F} \in \Lambda$ and if we put

$$\Phi'_{i,G,F} := \Phi_{i,G,F} \circ \Lambda_{i,F} : K_i(\phi)^p \times K_{i+1}(\phi)^q \longrightarrow K_i(F \otimes G)$$

then $\{\Phi'_{i,G,F}\} \in \vec{G}$.

b) If $\{\Phi'_{i,G,F}\} \in \vec{G}$ and if we put

$$\Lambda_{i,F} := \Phi_{i,G,F}^{-1} \circ \Phi'_{i,G,F} : K_i(\phi)^p \times K_{i+1}(\phi)^q \longrightarrow K_i(\phi)^p \times K_{i+1}(\phi)^q$$

then $\{\Lambda_{i,F}\} \in \Lambda$.

c) If $\{\Lambda_{i,F}\}, \{\Lambda'_{i,F}\} \in \Lambda$ then $\{\Lambda_{i,F} \circ \Lambda'_{i,F}\} \in \Lambda$, $\{\Lambda_{i,F}^{-1}\} \in \Lambda$. ■

DEFINITION 1.5.3 We denote for every nuclear $G \in \Upsilon$ by G_Υ the class of exact sequences in \mathfrak{M}_E

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

such that if δ_i denote its index maps then the diagram

$$\begin{array}{ccc} K_i(F_3)^{p(G)} \times K_{i+1}(F_3)^{q(G)} & \xrightarrow{\Phi_{i,G,F_3}} & K_i(F_3 \otimes G) \\ \delta_i^{p(G)} \times \delta_{i+1}^{q(G)} \downarrow & & \downarrow \delta_{G,i} \\ K_{i+1}(F_1)^{p(G)} \times K_i(F_1)^{q(G)} & \xrightarrow{\Phi_{(i+1),G,F_1}} & K_{i+1}(F_1 \otimes G) \end{array}$$

is commutative.

If G is Υ -null then every exact sequence in \mathfrak{M}_E belongs to G_Υ .

PROPOSITION 1.5.4

a) 0 is Υ -null.

b) $\mathbf{C} \in \Upsilon$, $p(\mathbf{C}) = 1$, $q(\mathbf{C}) = 0$, $\Phi_{i,\mathbf{C},F} = K_i(\phi_{\mathbf{C},F})$, where

$$\phi_{\mathbf{C},F} : F \longrightarrow F \times \mathbf{C}, \quad x \longmapsto x \otimes 1_{\mathbf{C}}.$$

Every exact sequence in \mathfrak{M}_E belongs to \mathbf{C}_Υ .

c) Let $G \xrightarrow{\varphi} G'$, $G' \xrightarrow{\psi} G$ be a homotopy in $\mathfrak{M}_{\mathbb{C}}$. If $G \in \Upsilon$ then

$$G' \in \Upsilon, \quad p(G') = p(G), \quad q(G') = q(G),$$

$$\Phi_{i,G',F} = K_i(id_F \otimes \varphi) \circ \Phi_{i,G,F}.$$

If in addition G and G' are nuclear then $G_{\Upsilon} = G'_{\Upsilon}$.

d) If G is null-homotopic then G is Υ -null.

a) By the null-axiom (Axiom 1.2.2), 0 is Υ -null.

b) The first assertion is easy to see. The second one follows from the commutativity of the index maps (Axiom 1.2.8).

c) By Proposition 1.4.2 b),

$$F \otimes G \xrightarrow{id_F \otimes \varphi} F \otimes G', \quad F \otimes G' \xrightarrow{id_F \otimes \psi} F \otimes G$$

is a homotopy in \mathfrak{M}_E . By Proposition 1.3.7 a),

$$K_i(id_F \otimes \varphi) : K_i(F \otimes G) \longrightarrow K_i(F \otimes G'),$$

$$K_i(id_F \otimes \psi) : K_i(F \otimes G') \longrightarrow K_i(F \otimes G)$$

are group isomorphisms and $K_i(id_F \otimes \psi) = K_i(id_F \otimes \varphi)^{-1}$. Thus

$$K_i(id_F \otimes \varphi) \circ \Phi_{i,G,F} : K_i(F)^{p(G)} \times K_{i+1}(F)^{q(G)} \longrightarrow K_i(F \otimes G')$$

is a group isomorphism. If $F \xrightarrow{\phi} F'$ is a morphism in \mathfrak{M}_E then the diagram

$$\begin{array}{ccccc} K_i(F)^{p(G)} \times K_{i+1}(F)^{q(G)} & \xrightarrow{\Phi_{i,G,F}} & K_i(F \otimes G) & \xrightarrow{K_i(id_F \otimes \varphi)} & K_i(F \otimes G') \\ \downarrow K_i(\phi)^{p(G)} \times K_{i+1}(\phi)^{q(G)} & & \downarrow K_i(\phi \otimes id_G) & & K_i(\phi \otimes id_{G'}) \downarrow \\ K_i(F')^{p(G)} \times K_{i+1}(F')^{q(G)} & \xrightarrow{\Phi_{i,G,F'}} & K_i(F' \otimes G) & \xrightarrow{K_i(id_{F'} \otimes \varphi)} & K_i(F' \otimes G') \end{array}$$

is commutative and the first assertion follows.

Assume now that G and G' are nuclear, let

$$(0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0) \in G_{\Upsilon},$$

and let δ_i be its associated index maps. By the commutativity of the index maps (Axiom 1.2.8 a)) the diagram

$$\begin{array}{ccc}
 K_i(F_3)^{p(G)} \times K_{i+1}(F_3)^{q(G)} & \xrightarrow{\delta_i^{p(G)} \times \delta_{i+1}^{q(G)}} & K_{i+1}(F_1)^{p(G)} \times K_i(F_1)^{q(G)} \\
 \Phi_{i,G,F_3} \downarrow & & \downarrow \Phi_{(i+1),G,F_1} \\
 K_i(F_3 \otimes G) & \xrightarrow{\delta_{G,i}} & K_{i+1}(F_1 \otimes G) \\
 K_i(id_{F_3} \otimes \varphi) \downarrow & & \downarrow K_{i+1}(id_{F_1} \otimes \varphi) \\
 K_i(F_3 \otimes G') & \xrightarrow{\delta_{G',i}} & K_{i+1}(F_1 \otimes G')
 \end{array}$$

is commutative. Since the maps of the columns are group isomorphisms, it follows by the above, that the diagram

$$\begin{array}{ccc}
 K_i(F_3)^{p(G')} \times K_{i+1}(F_3)^{q(G')} & \xrightarrow{\delta_i^{p(G')} \times \delta_{i+1}^{q(G')}} & K_{i+1}(F_1)^{p(G')} \times K_i(F_1)^{q(G')} \\
 \Phi_{i,G',F_3} \downarrow & & \downarrow \Phi_{(i+1),G',F_1} \\
 K_i(F_3 \otimes G') & \xrightarrow{\delta_{G',i}} & K_{i+1}(F_1 \otimes G')
 \end{array}$$

is also commutative.

d) follows from a) and c). ■

PROPOSITION 1.5.5 *Let G be a nuclear C^* -algebra belonging to Υ .*

a) *Every split exact sequence in \mathfrak{M}_E belongs to G_Υ .*

b) *Every exact sequence in \mathfrak{M}_E*

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

for which F_1 or F_3 is homotopic to 0 belongs to G_Υ .

a) Let

$$0 \longrightarrow F_1 \xrightarrow{\varphi} F_2 \xrightarrow[\lambda]{\psi} F_3 \longrightarrow 0$$

be a split exact sequence in \mathfrak{M}_E and let δ_i be its index maps. By Proposition 1.4.8 a),

$$0 \longrightarrow F_1 \otimes G \xrightarrow{\varphi \otimes id_G} F_2 \otimes G \xrightleftharpoons[\lambda \otimes id_G]{\psi \otimes id_G} F_3 \otimes G \longrightarrow 0$$

is split exact and so by Proposition 1.3.1, $\delta_i = \delta_{G,i} = 0$.

b) By Proposition 1.4.2 c), $F_1 \otimes G$ or $F_3 \otimes G$ is null-homotopic and so K-null. Thus by the six-term axiom (Axiom 1.2.7), $\delta_i = \delta_{G,i} = 0$, where δ_i denote the index maps associated to

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0 . \quad \blacksquare$$

PROPOSITION 1.5.6 *Let*

$$0 \longrightarrow G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\psi} G_3 \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_{\mathbf{C}}$ such that G_3 is nuclear.

a) Assume G_1 is Υ -null.

a₁) $K_i(id_F \otimes \psi) : K_i(F \otimes G_2) \longrightarrow K_i(F \otimes G_3)$ is a group isomorphism.

a₂) If $G_2 \in \Upsilon$ or $G_3 \in \Upsilon$ then

$$G_2, G_3 \in \Upsilon, \quad p(G_2) = p(G_3), \quad q(G_2) = q(G_3),$$

$$\Phi_{i,G_3,F} = K_i(id_F \otimes \psi) \circ \Phi_{i,G_2,F} .$$

If in addition G_2 is nuclear then $(G_2)_{\Upsilon} = (G_3)_{\Upsilon}$.

b) Assume G_2 is Υ -null and let δ_i^F denote the index maps of the exact sequence in \mathfrak{M}_E

$$0 \longrightarrow F \otimes G_1 \xrightarrow{id_F \otimes \varphi} F \otimes G_2 \xrightarrow{id_F \otimes \psi} F \otimes G_3 \longrightarrow 0 .$$

b₁) $\delta_i^F : K_i(F \otimes G_3) \longrightarrow K_{i+1}(F \otimes G_1)$ is a group isomorphism.

b₂) If $G_1 \in \Upsilon$ or $G_3 \in \Upsilon$ then

$$G_1, G_3 \in \Upsilon, \quad p(G_1) = q(G_3), \quad q(G_1) = p(G_3),$$

$$\Phi_{i,G_3,F} = \Phi_{(i+1),G_1,F} \circ \delta_i^F .$$

c) Assume G_3 is Υ -null.

$c_1)$ $K_i(id_F \otimes \varphi) : K_i(F \otimes G_1) \longrightarrow K_i(F \otimes G_2)$ is a group isomorphism.

$c_2)$ If $G_1 \in \Upsilon$ or $G_2 \in \Upsilon$ then

$$G_1, G_2 \in \Upsilon, \quad p(G_1) = p(G_2), \quad q(G_1) = q(G_2),$$

$$\Phi_{i,G_2,F} = K_i(id_F \otimes \varphi) \circ \Phi_{i,G_1,F}.$$

If in addition G_1 and G_2 are nuclear then $(G_1)_\Upsilon = (G_2)_\Upsilon$.

By Proposition 1.4.9 a), the sequence in \mathfrak{M}_E

$$0 \longrightarrow F \otimes G_1 \xrightarrow{id_F \otimes \varphi} F \otimes G_2 \xrightarrow{id_F \otimes \psi} F \otimes G_3 \longrightarrow 0$$

is exact. If G_j is Υ -null then $F \otimes G_j$ is K -null so $a_1), b_1), c_1)$ follow from Proposition 1.3.6 a), b).

$a_2)$ By $a_1)$, it is easy to see that

$$G_2, G_3 \in \Upsilon, \quad p(G_2) = p(G_3), \quad q(G_2) = q(G_3),$$

$$\Phi_{i,G_3,F} = K_i(id_F \otimes \psi) \circ \Phi_{i,G_2,F}.$$

Assume now G_2 nuclear. Let

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

belong to $(G_2)_\Upsilon$ or $(G_3)_\Upsilon$ and let δ_i be its associated index maps. Consider the diagram

$$\begin{array}{ccc} K_i(F_3)^{p(G_2)} \times K_{i+1}(F_3)^{q(G_2)} & \xrightarrow{\delta_i^{p(G_2)} \times \delta_{i+1}^{q(G_2)}} & K_{i+1}(F_1)^{p(G_2)} \times K_i(F_1)^{q(G_2)} \\ \Phi_{i,G_2,F_3} \downarrow & & \downarrow \Phi_{(i+1),G_2,F_1} \\ K_i(F_3 \otimes G_2) & \xrightarrow{\delta_{G_2,i}} & K_{i+1}(F_1 \otimes G_2) \\ K_i(id_{F_3} \otimes \psi) \downarrow & & \downarrow K_{i+1}(id_{F_1} \otimes \psi) \\ K_i(F_3 \otimes G_3) & \xrightarrow{\delta_{G_3,i}} & K_{i+1}(F_1 \otimes G_3) \\ \Phi_{i,G_3,F_3} \uparrow & & \uparrow \Phi_{(i+1),G_3,F_1} \\ K_i(F_3)^{p(G_3)} \times K_{i+1}(F_3)^{q(G_3)} & \xrightarrow{\delta_i^{p(G_3)} \times \delta_{i+1}^{q(G_3)}} & K_{i+1}(F_1)^{p(G_3)} \times K_i(F_1)^{q(G_3)}. \end{array}$$

Its upper part or lower part is commutative and the maps of the columns are group isomorphisms. It follows, by the above, that the diagram is commutative. Thus $(G_2)_\Upsilon = (G_3)_\Upsilon$.

$b_2)$ Let $F \xrightarrow{\phi} F'$ be a morphism in \mathfrak{M}_E . Then the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K_i(F \otimes G_1) & \xrightarrow{K_i(id_F \otimes \phi)} & K_i(F \otimes G_2) & \xrightarrow{K_i(id_F \otimes \psi)} & \\
 & & \downarrow K_i(\phi \otimes id_{G_1}) & & \downarrow K_i(\phi \otimes id_{G_2}) & & \\
 0 & \longrightarrow & K_i(F' \otimes G_1) & \xrightarrow{K_i(id_{F'} \otimes \phi)} & K_i(F' \otimes G_2) & \xrightarrow{K_i(id_{F'} \otimes \psi)} & \\
 & & & & & & \\
 & & \xrightarrow{K_i(id_F \otimes \phi)} & K_i(F \otimes G_2) & \xrightarrow{K_i(id_F \otimes \psi)} & K_i(F \otimes G_3) & \longrightarrow 0 \\
 & & \downarrow K_i(\phi \otimes id_{G_2}) & & \downarrow K_i(\phi \otimes id_{G_3}) & & \\
 & & \xrightarrow{K_i(id_{F'} \otimes \phi)} & K_i(F' \otimes G_2) & \xrightarrow{K_i(id_{F'} \otimes \psi)} & K_i(F' \otimes G_3) & \longrightarrow 0
 \end{array}$$

is commutative and has exact rows. By the commutativity of the index maps (Axiom 1.2.8), the diagram

$$\begin{array}{ccc}
 K_i(F \otimes G_3) & \xrightarrow{\delta_i^F} & K_{i+1}(F \otimes G_1) \\
 \downarrow K_i(\phi \otimes id_{G_3}) & & \downarrow K_{i+1}(\phi \otimes id_{G_1}) \\
 K_i(F' \otimes G_3) & \xrightarrow{\delta_i^{F'}} & K_{i+1}(F' \otimes G_1)
 \end{array}$$

is commutative. By $b_1)$,

$$G_1, G_3 \in \Upsilon, \quad p(G_1) = q(G_3), \quad q(G_1) = p(G_3),$$

$$\Phi_{i,G_3,F} = \Phi_{(i+1),G_1,F} \circ \delta_i^F.$$

$c_2)$ The proof is similar to the proof of $a_2)$. ■

PROPOSITION 1.5.7 *Let*

$$0 \longrightarrow G_1 \xrightarrow{\phi} G_2 \xrightarrow[\lambda]{\psi} G_3 \longrightarrow 0$$

be a split exact sequence in $\mathfrak{M}_\mathbf{C}$.

a) If $G_1, G_3 \in \Upsilon$ then

$$G_2 \in \Upsilon, \quad p(G_2) = p(G_1) + p(G_3), \quad q(G_2) = q(G_1) + q(G_3),$$

$$\Phi_{i,G_2,F} = (K_i(id_F \otimes \varphi) \times K_i(id_F \otimes \lambda)) \circ (\Phi_{i,G_1,F} \times \Phi_{i,G_3,F}).$$

b) If in addition G_1, G_2 , and G_3 are nuclear then $(G_1)_\Upsilon \cap (G_3)_\Upsilon \subset (G_2)_\Upsilon$.

a) By Proposition 1.4.3 b), the sequence

$$0 \longrightarrow K_i(F \otimes G_1) \xrightarrow{K_i(id_F \otimes \varphi)} K_i(F \otimes G_2) \xrightleftharpoons[K_i(id_F \otimes \lambda)]{K_i(id_F \otimes \psi)} K_i(F \otimes G_3) \longrightarrow 0$$

is split exact. Thus the maps

$$\begin{aligned} & \left(K_i(F)^{p(G_1)} \times K_{i+1}(F)^{q(G_1)} \right) \times \left(K_i(F)^{p(G_3)} \times K_{i+1}(F)^{q(G_3)} \right) \\ & \xrightarrow{\Phi_{i,G_1,F} \times \Phi_{i,G_3,F}} K_i(F \otimes G_1) \times K_i(F \otimes G_3) \xrightarrow{K_i(id_F \otimes \varphi) \times K_i(id_F \otimes \lambda)} K_i(F \otimes G_2) \end{aligned}$$

are group isomorphisms.

Let $F \xrightarrow{\phi} F'$ be a morphism in \mathfrak{M}_E . Since the diagram with split exact rows

$$\begin{aligned} 0 \longrightarrow F \otimes G_1 & \xrightarrow{id_F \otimes \varphi} F \otimes G_2 \xrightleftharpoons[id_F \otimes \lambda]{id_F \otimes \psi} F \otimes G_3 \longrightarrow 0, \\ 0 \longrightarrow F' \otimes G_1 & \xrightarrow{id_{F'} \otimes \varphi} F' \otimes G_2 \xrightleftharpoons[id_{F'} \otimes \lambda]{id_{F'} \otimes \psi} F' \otimes G_3 \longrightarrow 0, \end{aligned}$$

(Proposition 1.4.3 a)) and with columns $\phi \otimes id_{G_1}$, $\phi \otimes id_{G_2}$, and $\phi \otimes id_{G_3}$ is commutative, the assertion follows from Proposition 1.4.3 b).

b) Let

$$(0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0) \in (G_1)_\Upsilon \cap (G_3)_\Upsilon$$

and let δ_i be its associated index maps. Consider the diagram (by a))

$$\begin{array}{ccc}
 K_i(F_3)^{p(G_2)} \times K_{i+1}(F_3)^{q(G_2)} & \xrightarrow{\delta_i^{p(G_2)} \times \delta_{i+1}^{q(G_2)}} & K_{i+1}(F_1)^{p(G_2)} \times K_i(F_1)^{q(G_2)} \\
 \Phi_{i,G_1,F_3} \times \Phi_{i,G_3,F_3} \downarrow & & \Phi_{(i+1),G_1,F_1} \times \Phi_{(i+1),G_3,F_1} \downarrow \\
 K_i(F_3 \otimes G_1) \times K_i(F_3 \otimes G_3) & \xrightarrow{\delta_{G_1,i} \times \delta_{G_3,i}} & K_{i+1}(F_1 \otimes G_1) \times K_{i+1}(F_1 \otimes G_3) \\
 A \downarrow & & K_{i+1}(id_{F_1} \otimes \varphi) \times K_{i+1}(id_{F_1} \otimes \lambda) \downarrow \\
 K_i(F_3 \otimes G_2) & \xrightarrow{\delta_{G_2,i}} & K_{i+1}(F_1 \otimes G_2) \\
 \Phi_{i,G_2,F_3} \uparrow & & \uparrow \Phi_{(i+1),G_2,F_1} \\
 K_i(F_3)^{p(G_2)} \times K_{i+1}(F_3)^{q(G_2)} & \xrightarrow{\delta_i^{p(G_2)} \times \delta_{i+1}^{q(G_2)}} & K_{i+1}(F_1)^{p(G_2)} \times K_i(F_1)^{q(G_2)},
 \end{array}$$

where

$$A := K_i(id_{F_3} \otimes \varphi) \times K_i(id_{F_3} \otimes \lambda) .$$

Its upper part is commutative and the maps of the columns are group isomorphisms. It follows that the lower part of the diagram is also commutative. ■

COROLLARY 1.5.8 *If $G \in \Upsilon$ then $\tilde{G} \in \Upsilon$, $p(\tilde{G}) = p(G) + 1$, $q(\tilde{G}) = q(G)$. If in addition G and \tilde{G} are nuclear then $G_\Upsilon \subset \tilde{G}_\Upsilon$.* ■

PROPOSITION 1.5.9 *Let $(G_j)_{j \in J}$ be a finite family in Υ .*

a)

$$G := \prod_{j \in J} G_j \in \Upsilon, \quad p(G) = \sum_{j \in J} p(G_j), \quad q(G) = \sum_{j \in J} q(G_j),$$

$$\Phi_{i,G,F} = \left(\prod_{j \in J} \Phi_{i,G_j,F} \right) \circ \Phi_{(F \otimes G_j)_{j \in J}, i} .$$

In particular if G_j is Υ -null for every $j \in J$ then G is Υ -null.

b) *If in addition G and all G_j , $j \in J$, are nuclear then*

$$\bigcap_{j \in J} (G_j)_\Upsilon \subset G_\Upsilon .$$

c) $\mathbf{C}^J \in \Upsilon$, $p(\mathbf{C}^J) = \text{Card } J$, $q(\mathbf{C}^J) = 0$, and every exact sequence in \mathfrak{M}_E belongs to $(\mathbf{C}^J)_\Upsilon$.

a) We put

$$\bar{p} := \sum_{j \in J} p(G_j), \quad \bar{q} := \sum_{j \in J} q(G_j).$$

Since

$$F \otimes \prod_{j \in J} G_j \approx \prod_{j \in J} (F \otimes G_j),$$

by Proposition 1.3.3, the maps

$$\begin{aligned} \prod_{j \in J} \Phi_{i, G_j, F} : K_i(F)^{\bar{p}} \times K_{i+1}(F)^{\bar{q}} &= \prod_{j \in J} \left(K_i(F)^{p(G_j)} \times K_{i+1}(F)^{q(G_j)} \right) \longrightarrow \\ &\longrightarrow \prod_{j \in J} K_i(F \otimes G_j) \xrightarrow{\Phi_{(F \otimes G_j)_{j \in J, i}}} K_i(F \otimes G) \end{aligned}$$

are group isomorphisms. Let $F \xrightarrow{\phi} F'$ be a morphism in \mathfrak{M}_E . The diagram

$$\begin{array}{ccc} K_i(F)^{\bar{p}} \times K_{i+1}(F)^{\bar{q}} & \xrightarrow{\prod_{j \in J} \Phi_{i, G_j, F}} & \prod_{j \in J} K_i(F \otimes G_j) \\ K_i(\phi)^{\bar{p}} \times K_{i+1}(\phi)^{\bar{q}} \downarrow & & \downarrow \prod_{j \in J} K_i(\phi \otimes id_{G_j}) \\ K_i(F')^{\bar{p}} \times K_{i+1}(F')^{\bar{q}} & \xrightarrow{\prod_{j \in J} \Phi_{i, G_j, F'}} & \prod_{j \in J} K_i(F' \otimes G_j) \end{array}$$

is obviously commutative and by Proposition 1.3.4 the diagram

$$\begin{array}{ccc} \prod_{j \in J} K_i(F \otimes G_j) & \xrightarrow{\Phi_{(F \otimes G_j)_{j \in J, i}}} & K_i(F \otimes G) \\ \prod_{j \in J} K_i(\phi \otimes id_{G_j}) \downarrow & & \downarrow K_i(\phi \otimes id_G) \\ \prod_{j \in J} K_i(F' \otimes G_j) & \xrightarrow{\Phi_{(F' \otimes G_j)_{j \in J, i}}} & K_i(F' \otimes G) \end{array}$$

is also commutative and this proves the assertion.

b) follows from Proposition 1.5.7 by complete induction.

c) follows from a), b), and Proposition 1.5.4 b). ■

PROPOSITION 1.5.10 *Let J be a finite set and for every $j \in J$ let*

$$0 \longrightarrow F_{j,1} \xrightarrow{\varphi_j} F_{j,2} \xrightarrow{\psi_j} F_{j,3} \longrightarrow 0$$

be an exact sequence in \mathfrak{M}_E and $\delta_{j,i}$ its associated index maps. For every $k \in \{1, 2, 3\}$ put

$$F_k := \prod_{j \in J} F_{j,k}$$

and for every $j \in J$ denote by

$$\varphi_{j,k} : F_{j,k} \longrightarrow F_k, \quad \psi_{j,k} : F_k \longrightarrow F_{j,k}$$

the canonical inclusion and projection, respectively. Then

$$0 \longrightarrow F_1 \xrightarrow{\prod_{j \in J} \varphi_j} F_2 \xrightarrow{\prod_{j \in J} \psi_j} F_3 \longrightarrow 0$$

is an exact sequence in \mathfrak{M}_E and if we denote by δ_i its index maps then the diagram

$$\begin{array}{ccc} \prod_{j \in J} K_i(F_{j,3}) & \xrightarrow{\Psi_{3,i}} & K_i(F_3) \\ \prod_{j \in J} \delta_{j,i} \downarrow & & \downarrow \delta_i \\ \prod_{j \in J} K_{i+1}(F_{j,1}) & \xrightarrow{\Psi_{1,(i+1)}} & K_{i+1}(F_1) \end{array}$$

is commutative, where for every $k \in \{1, 3\}$,

$$\Psi_{k,i} : \prod_{j \in J} K_i(F_{j,k}) \longrightarrow K_i(F_k), \quad (a_j)_{j \in J} \longmapsto \sum_{j \in J} K_i(\varphi_{j,k}) a_j.$$

For every $j \in J$ the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_{j,1} & \xrightarrow{\varphi_j} & F_{j,2} & \xrightarrow{\psi_j} & F_{j,3} \longrightarrow 0 \\ & & \varphi_{j,1} \downarrow & & \varphi_{j,2} \downarrow & & \downarrow \varphi_{j,3} \\ 0 & \longrightarrow & F_1 & \xrightarrow{\prod_{j \in J} \varphi_j} & F_2 & \xrightarrow{\prod_{j \in J} \psi_j} & F_3 \longrightarrow 0, \end{array}$$

is commutative. By the commutativity of the index maps (Axiom 1.2.8), the diagram

$$\begin{array}{ccc} K_i(F_{j,3}) & \xrightarrow{K_i(\varphi_{j,3})} & K_i(F_3) \\ \delta_{j,i} \downarrow & & \downarrow \delta_i \\ K_{i+1}(F_{j,1}) & \xrightarrow{K_{i+1}(\varphi_{j,1})} & K_{i+1}(F_1) \end{array}$$

is commutative. Let $(a_j)_{j \in J} \in \prod_{j \in J} K_i(F_{j,3})$. Then

$$\begin{aligned} \delta_i \Psi_{3,i}(a_j)_{j \in J} &= \delta_i \sum_{j \in J} K_i(\varphi_{j,3}) a_j = \\ &= \sum_{j \in J} K_{i+1}(\varphi_{j,1}) \delta_{j,i} a_j = \Psi_{1,(i+1)} \left(\prod_{j \in J} \delta_{j,i} \right) (a_j)_{j \in J}. \end{aligned}$$

Thus the diagram

$$\begin{array}{ccc} \prod_{j \in J} K_i(F_{j,3}) & \xrightarrow{\Psi_{3,i}} & K_i(F_3) \\ \prod_{j \in J} \delta_{j,i} \downarrow & & \downarrow \delta_i \\ \prod_{j \in J} K_{i+1}(F_{j,1}) & \xrightarrow{\Psi_{1,(i+1)}} & K_{i+1}(F_1) \end{array}$$

is commutative. ■

PROPOSITION 1.5.11 *Let $(G_j)_{j \in J}$ be a finite family in Υ .*

a)

$$G := \bigotimes_{j \in J} G_j \in \Upsilon,$$

$$\begin{aligned} p(G) &= \frac{1}{2} \left(\prod_{j \in J} (p(G_j) + q(G_j)) + \prod_{j \in J} (p(G_j) - q(G_j)) \right), \\ q(G) &= \frac{1}{2} \left(\prod_{j \in J} (p(G_j) + q(G_j)) - \prod_{j \in J} (p(G_j) - q(G_j)) \right). \end{aligned}$$

b) *If G_{j_0} is K -null for a $j_0 \in J$ then $F \otimes \left(\bigotimes_{j \in J} G_j \right)$ is also K -null.*

c) *If $p(G_{j_0}) = q(G_{j_0})$ for a $j_0 \in J$ then $p(G) = q(G)$.*

d) *Let $j_0 \in J$, $J' := J \setminus \{j_0\}$, and $G' := \bigotimes_{j \in J'} G_j$.*

d₁) *If $p(G_{j_0}) = 1$, $q(G_{j_0}) = 0$ then $p(G') = p(G)$, $q(G') = q(G)$.*

d₂) *If $p(G_{j_0}) = 0$, $q(G_{j_0}) = 1$ then $p(G') = q(G)$, $q(G') = p(G)$.*

e) If we put

$$H := \bigotimes_{j \in J} \tilde{G}_j \quad \text{and} \quad G_I := \bigotimes_{j \in I} G_j$$

for every $I \subset J$ then

$$H \in \Upsilon, \quad p(H) = \sum_{I \subset J} p(G_I), \quad q(H) = \sum_{I \subset J} q(G_I);.$$

f) If in addition G and all $(G_j)_{j \in J}$ are nuclear then

$$\bigcap_{j \in J} (G_j)_{\Upsilon} \subset G_{\Upsilon}.$$

a) Assume first $J = \{1, 2\}$. The maps

$$\begin{aligned} & K_i(F)^{p(G_1)p(G_2)+q(G_1)q(G_2)} \times K_{i+1}(F)^{p(G_1)q(G_2)+p(G_2)q(G_1)} = \\ & = \left(K_i(F)^{p(G_1)} \times K_{i+1}(F)^{q(G_1)} \right)^{p(G_2)} \times \left(K_{i+1}(F)^{p(G_1)} \times K_i(F)^{q(G_1)} \right)^{q(G_2)} \longrightarrow \\ & \quad \xrightarrow{(\Phi_{i,G_1,F})^{p(G_2)} \times (\Phi_{(i+1),G_1,F})^{q(G_2)}} \\ & \longrightarrow K_i(F \otimes G_1)^{p(G_2)} \times K_{i+1}(F \otimes G_1)^{q(G_2)} \longrightarrow \\ & \quad \xrightarrow{\Phi_{i,G_2,F \otimes G_1}} \\ & \longrightarrow K_i((F \otimes G_1) \otimes G_2) \approx K_i(F \otimes (G_1 \otimes G_2)) \end{aligned}$$

are group isomorphisms and

$$\begin{aligned} p(G_1 \otimes G_2) &:= p(G_1)p(G_2) + q(G_1)q(G_2) = \\ &= \frac{1}{2}[(p(G_1) + q(G_1))(p(G_2) + q(G_2)) + (p(G_1) - q(G_1))(p(G_2) - q(G_2))], \\ q(G_1 \otimes G_2) &:= p(G_1)q(G_2) + p(G_2)q(G_1) = \\ &= \frac{1}{2}[(p(G_1) + q(G_1))(p(G_2) + q(G_2)) - (p(G_1) - q(G_1))(p(G_2) - q(G_2))]. \end{aligned}$$

If $F \xrightarrow{\phi} F'$ is a morphism in \mathfrak{M}_E then the diagrams

$$\begin{array}{ccc} K_i(F \otimes G_1)^{p(G_2)} \times K_{i+1}(F \otimes G_1)^{q(G_2)} & \xrightarrow{\Phi_{i,G_2,(F \otimes G_1)}} & K_i((F \otimes G_1) \otimes G_2) \\ \downarrow K_i(\phi \otimes id_{G_1})^{p(G_2)} \times K_{i+1}(\phi \otimes id_{G_1})^{q(G_2)} & & \downarrow K_i((\phi \otimes id_{G_1}) \otimes id_{G_2}) \\ K_i(F' \otimes G_1)^{p(G_2)} \times K_{i+1}(F' \otimes G_1)^{q(G_2)} & \xrightarrow{\Phi_{i,G_2,(F' \otimes G_1)}} & K_i((F' \otimes G_1) \otimes G_2) \end{array}$$

$$\begin{array}{ccc}
K_i((F \otimes G_1) \otimes G_2) & \xrightarrow{\approx} & K_i(F \otimes (G_1 \otimes G_2)) \\
\downarrow K_i((\phi \otimes id_{G_1}) \otimes id_{G_2}) & & \downarrow K_i(\phi \otimes id_{(G_1 \otimes G_2)}) \\
K_i((F' \otimes G_1) \otimes G_2) & \xrightarrow{\approx} & K_i(F' \otimes (G_1 \otimes G_2))
\end{array}$$

are commutative, which proves the assertion in this case.

The general case is obtained now by induction with respect to $Card J$. Let $Card J > 1$, $k \in J$, $J' := J \setminus \{k\}$, $G' := \otimes_{j \in J'} G_j$ and assume the assertion holds for J' . By the above,

$$\begin{aligned}
p(G) &= \frac{1}{2}[(p(G') + q(G'))(p(G_k) + q(G_k)) + (p(G') - q(G'))(p(G_k) - q(G_k))] = \\
&= \frac{1}{2} \left(\prod_{j \in J'} (p(G_j) + q(G_j))(p(G_k) + q(G_k)) + \right. \\
&\quad \left. + \prod_{j \in J'} (p(G_j) - q(G_j))(p(G_k) - q(G_k)) \right) = \\
&= \frac{1}{2} \left(\prod_{j \in J} (p(G_j) + q(G_j)) + \prod_{j \in J} (p(G_j) - q(G_j)) \right), \\
q(G) &= \frac{1}{2}[(p(G') + q(G'))(p(G_k) + q(G_k)) - (p(G') - q(G'))(p(G_k) - q(G_k))] = \\
&= \frac{1}{2} \left(\prod_{j \in J'} (p(G_j) + q(G_j))(p(G_k) + q(G_k)) - \right. \\
&\quad \left. - \prod_{j \in J'} (p(G_j) - q(G_j))(p(G_k) - q(G_k)) \right) = \\
&= \frac{1}{2} \left(\prod_{j \in J} (p(G_j) + q(G_j)) - \prod_{j \in J} (p(G_j) - q(G_j)) \right).
\end{aligned}$$

b), c), and d) follow directly from a).

e) By Corollary 1.5.8, $\tilde{G}_j \in \Upsilon$ for every $j \in J$. By a) and Proposition 1.4.6, $H \in \Upsilon$,

$$\begin{aligned}
K_i(F \otimes H) &\approx \prod_{I \subset J} K_i(F \otimes G_I) \approx \prod_{I \subset J} \left(K_i(F)^{p(G_I)} \times K_{i+1}(F)^{q(G_I)} \right) = \\
&= K_i(F)^{\sum_{I \subset J} p(G_I)} \times K_{i+1}(F)^{\sum_{I \subset J} q(G_I)}.
\end{aligned}$$

f) Assume first $J := \{1, 2\}$, let

$$(0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0) \in (G_1)_Y \cap (G_2)_Y,$$

and let δ_i be its index maps. Then (by a)) the diagram

$$\begin{array}{ccc} K_i(F_3)^{p(G)} \times K_{i+1}(F_3)^{q(G)} & \xrightarrow{\delta_i^{p(G)} \times \delta_{i+1}^{q(G)}} & A \\ \Phi_{i,G_1,F_3}^{p(G_2)} \times \Phi_{(i+1),G_1,F_3}^{q(G_2)} \downarrow & & \Phi_{(i+1),G_1,F_1}^{p(G_2)} \times \Phi_{i,G_1,F_1}^{q(G_2)} \downarrow \\ K_i(F_3 \otimes G_1)^{p(G_2)} \times K_{i+1}(F_3 \otimes G_1)^{q(G_2)} & \xrightarrow{\delta_{G_1,i}^{p(G_1)} \times \delta_{G_1,i}^{q(G_2)}} & B \\ \Phi_{i,G_2,(F_3 \otimes G_1)} \downarrow & & \Phi_{(i+1),G_2,(F_1 \otimes G_1)} \downarrow \\ K_i((F_3 \otimes G_1) \otimes G_2) \approx K_i(F_3 \otimes (G_1 \otimes G_2)) & \xrightarrow{\delta_{G_1,(G_2,i)}} & C \end{array}$$

is commutative, where

$$\begin{aligned} A &:= K_{i+1}(F_1)^{p(G)} \times K_i(F_1)^{q(G)}, \\ B &:= K_{i+1}(F_1 \otimes G_1)^{p(G_2)} \times K_i(F_1 \otimes G_1)^{q(G_2)}, \\ C &:= K_{i+1}(((F_1 \otimes G_1) \otimes G_2)) \approx K_{i+1}((F_1 \otimes (G_1 \otimes G_2))). \end{aligned}$$

Thus

$$(0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0) \in G_Y.$$

The general case follows by induction with respect to $\text{Card } J$. ■

COROLLARY 1.5.12 *Let $G \in Y$, $n \in \mathbb{N}$, and $H := \otimes_{j \in \mathbb{N}_n} G$. Then $H \in Y$, $G_Y \subset H_Y$, and*

$$\begin{aligned} p(H) &= \frac{1}{2} ((p(G) + q(G))^n + (p(G) - q(G))^n), \\ q(H) &= \frac{1}{2} ((p(G) + q(G))^n - (p(G) - q(G))^n). \end{aligned}$$

The assertion follows from Proposition 1.5.11 a). ■

PROPOSITION 1.5.13 *Let (G_1, G_2, G_3) be an \mathfrak{M}_T -triple such that G_1/G_3 and G_2/G_3 are nuclear, G_2 is Y -null, and $G_1, G_3 \in Y$. We use the notation of the triple theorem (Theorem 1.3.8 a)) associated to the \mathfrak{M}_E -triple*

$$(F \otimes G_1, F \otimes G_2, F \otimes G_3)$$

(Proposition 1.4.9), put $\varphi := \varphi_{1,2}/(F \otimes G_3)$ (as in Proposition 1.3.7 a)), and denote by

$$\Psi_{F,i} : K_i(F \otimes G_1) \times K_{i+1}(F \otimes G_3) \longrightarrow K_i(F \otimes (G_1/G_3)),$$

$$(a, b) \longmapsto K_i(\psi_{1,3})a + \Phi_i b$$

the corresponding group isomorphism (Theorem 1.3.8 a₄), Proposition 1.4.9). Then

$$G_1/G_3 \in \Upsilon, \quad p(G_1/G_3) = p(G_1) + q(G_3), \quad q(G_1/G_3) = q(G_1) + p(G_3),$$

$$\Phi_{i,(G_1/G_3),F} = \Psi_{F,i} \circ (\Phi_{i,G_1,F} \times \Phi_{(i+1),G_3,F}).$$

Since $G_1, G_3 \in \Upsilon$, the map

$$\begin{aligned} \Psi_{F,i} \circ (\Phi_{i,G_1,F} \times \Phi_{(i+1),G_3,F}) : & \left(K_i(F)^{p(G_1)} \times K_{i+1}(F)^{q(G_1)} \right) \times \\ & \times \left(K_{i+1}(F)^{p(G_3)} \times K_i(F)^{q(G_3)} \right) \longrightarrow K_i(F \otimes (G_1/G_3)) \end{aligned}$$

is a group isomorphism. We put

$$\bar{p}(G_1/G_3) := p(G_1) + q(G_3), \quad \bar{q}(G_1/G_3) := q(G_1) + p(G_3),$$

$$\bar{\Phi}_{i,G_1/G_3,F} := \Psi_{F,i} \circ (\Phi_{i,G_1,F} \times \Phi_{(i+1),G_3,F}).$$

Let $F \xrightarrow{\phi} F'$ be a morphism in \mathfrak{M}_E . We mark with a prime the above notation associated to F' . By the commutativity of the index maps (Axiom 1.2.8),

$$K_{i+1}(\phi \otimes id_{G_3}) \circ \delta_{2,3,i} = \delta'_{2,3,i} \circ K_i(\phi \otimes id_{(G_2/G_3)}).$$

Moreover

$$K_i(\phi \otimes id_{(G_1/G_3)}) \circ K_i(\varphi) = K_i(\varphi') \circ K_i(\phi \otimes id_{(G_2/G_3)}),$$

$$K_i(\phi \otimes id_{(G_1/G_3)}) \circ K_i(\psi_{1,3}) = K_i(\psi'_{1,3}) \circ K_i(\phi \otimes id_{G_1}).$$

It follows

$$\begin{aligned} K_i(\phi \otimes id_{(G_1/G_3)}) \circ \Phi_i &= K_i(\phi \otimes id_{(G_1/G_3)}) \circ K_i(\varphi) \circ (\delta_{2,3,i})^{-1} = \\ &= K_i(\varphi') \circ K_i(\phi \otimes id_{(G_2/G_3)}) \circ (\delta_{2,3,i})^{-1} = \\ &= K_i(\varphi') \circ (\delta'_{2,3,i})^{-1} \circ K_{i+1}(\phi \otimes id_{G_3}) = \Phi'_i \circ K_{i+1}(\phi \otimes id_{G_3}). \end{aligned}$$

We want to prove that the diagram

$$\begin{array}{ccc}
 K_i(F \otimes G_1) \times K_{i+1}(F \otimes G_3) & \xrightarrow{\Psi_{F,i}} & K_i(F \otimes (G_1/G_3)) \\
 K_i(\phi \otimes id_{G_1}) \times K_{i+1}(\phi \otimes id_{G_3}) \downarrow & & \downarrow K_i(\phi \otimes id_{(G_1/G_3)}) \\
 K_i(F' \otimes G_1) \times K_{i+1}(F' \otimes G_3) & \xrightarrow{\Psi_{F',i}} & K_i(F' \otimes (G_1/G_3))
 \end{array}$$

is commutative. For $(a, b) \in K_i(F \otimes G_1) \times K_{i+1}(F \otimes G_3)$, by the above,

$$\begin{aligned}
 K_i(\phi \otimes id_{(G_1/G_3)}) \Psi_{F,i}(a, b) &= K_i(\phi \otimes id_{(G_1/G_3)}) (K_i(\psi_{1,3})a + \Phi_i b) = \\
 &= K_i(\phi \otimes id_{(G_1/G_3)}) K_i(\psi_{1,3})a + K_i(\phi \otimes id_{(G_1/G_3)}) \Phi_i b = \\
 &= K_i(\psi'_{1,3}) K_i(\phi \otimes id_{G_1})a + \Phi'_i K_{i+1}(\phi \otimes id_{G_3})b = \\
 &= \Psi_{F',i}(K_i(\phi \otimes id_{G_1})a, K_{i+1}(\phi \otimes id_{G_3})b) = \\
 &= \Psi_{F',i}(K_i(\phi \otimes id_{G_1}) \times K_{i+1}(\phi \otimes id_{G_3}))(a, b).
 \end{aligned}$$

Thus the above diagram is commutative. It follows, since $G_1, G_3 \in \Upsilon$, that the diagram

$$\begin{array}{ccc}
 K_i(F)^{p(G_1/G_3)} \times K_{i+1}(F)^{q(G_1/G_3)} & \xrightarrow{\Phi_{i,(G_1/G_3),F}} & K_i(F \otimes (G_1/G_3)) \\
 K_i(\phi)^{p(G_1/G_3)} \times K_{i+1}(\phi)^{q(G_1/G_3)} \downarrow & & \downarrow K_i(\phi \otimes id_{(G_1/G_3)}) \\
 K_i(F')^{p(G_1/G_3)} \times K_{i+1}(F')^{q(G_1/G_3)} & \xrightarrow{\Phi_{i,(G_1/G_3),F'}} & K_i(F' \otimes (G_1/G_3))
 \end{array}$$

is commutative. Hence

$$G_1/G_3 \in \Upsilon, \quad p(G_1/G_3) = p(G_1) + q(G_3), \quad q(G_1/G_3) = q(G_1) + p(G_3),$$

$$\Phi_{i,(G_1/G_3),F} = \Psi_{F,i} \circ (\Phi_{i,G_1,F} \times \Phi_{(i+1),G_3,F}). \quad \blacksquare$$

PROPOSITION 1.5.14 *Let (G_1, G_2, G_3) be an $\mathfrak{M}_{\mathbb{C}}$ -triple such that G_1/G_2 and G_1/G_3 are nuclear, G_1/G_3 is Υ -null, and $G_1, G_1/G_2 \in \Upsilon$. We use the notation of the triple theorem (Theorem 1.3.8 b)) associated to the \mathfrak{M}_E -triple*

$$(F \otimes G_1, F \otimes G_2, F \otimes G_3)$$

(Proposition 1.4.9), assume ψ_{12} K -null for all E - C^* -algebras F , and denote by

$$\Psi_{F,i} : K_i(F \otimes G_1) \times K_{i+1}(F \otimes (G_1/G_2)) \longrightarrow K_i(F \otimes G_2),$$

$$(a, b) \mapsto \Phi'_i a + \delta_{1,2,(i+1)} b$$

the corresponding group isomorphism (Theorem 1.3.8 b_4), Proposition 1.4.9). Then

$$G_2 \in \Upsilon, \quad p(G_2) = p(G_1) + q(G_1/G_2), \quad q(G_2) = q(G_1) + p(G_1/G_2),$$

$$\Phi_{i,G_2,F} = \Psi_{F,i} \circ (\Phi_{i,G_1,F} \times \Phi_{(i+1),(G_1/G_2),F}).$$

Since $G_1, G_1/G_2 \in \Upsilon$, the map

$$\begin{aligned} \Psi_{F,i} \circ (\Phi_{i,G_1,F} \times \Phi_{(i+1),(G_1/G_2),F}) : & \left(K_i(F)^{p(G_1)} \times K_{i+1}(F)^{q(G_1)} \right) \times \\ & \times \left(K_{i+1}(F)^{p(G_1/G_2)} \times K_i(F)^{q(G_1/G_2)} \right) \longrightarrow K_i(F \otimes G_2) \end{aligned}$$

is a group isomorphism. We put

$$\tilde{p}(G_2) := p(G_1) + q(G_1/G_2), \quad \tilde{q}(G_2) := q(G_1) + p(G_1/G_2),$$

$$\tilde{\Phi}_{i,G_2,F} := \Psi_{F,i} \circ (\Phi_{i,G_1,F} \times \Phi_{(i+1),(G_1/G_2),F}).$$

Let $F \xrightarrow{\phi} \bar{F}$ be a morphism in \mathfrak{M}_E . We mark with a bar the above notation associated to \bar{F} . By the commutativity of the index maps (Axiom 1.2.8),

$$K_i(\phi \otimes id_{G_2}) \circ \delta_{1,2,(i+1)} = \bar{\delta}_{1,2,(i+1)} \circ K_{i+1}(\phi \otimes id_{(G_1/G_2)}).$$

Moreover

$$K_i(\phi \otimes id_{G_1}) \circ K_i(\varphi_{1,3}) = K_i(\bar{\varphi}_{1,3}) \circ K_i(\phi \otimes id_{G_3}),$$

$$K_i(\phi \otimes id_{G_2}) \circ K_i(\varphi_{2,3}) = K_i(\bar{\varphi}_{2,3}) \circ K_i(\phi \otimes id_{G_3}).$$

It follows

$$\begin{aligned} K_i(\phi \otimes id_{G_2}) \circ \Phi'_i &= K_i(\phi \otimes id_{G_2}) \circ K_i(\varphi_{2,3}) \circ K_i(\varphi_{1,3})^{-1} = \\ &= K_i(\bar{\varphi}_{2,3}) \circ K_i(\phi \otimes id_{G_3}) \circ K_i(\varphi_{1,3})^{-1} = \\ &= K_i(\bar{\varphi}_{2,3}) \circ K_i(\bar{\varphi}_{1,3})^{-1} \circ K_i(\phi \otimes id_{G_1}) = \bar{\Phi}'_i \circ K_i(\phi \otimes id_{G_1}). \end{aligned}$$

We want to prove that the diagram

$$\begin{array}{ccc} K_i(F \otimes G_1) \times K_{i+1}(F \otimes (G_1/G_2)) & \xrightarrow{\Psi_{F,i}} & K_i(F \otimes G_2) \\ K_i(\phi \otimes id_{G_1}) \times K_{i+1}(\phi \otimes id_{(G_1/G_2)}) \downarrow & & \downarrow K_i(\phi \otimes id_{G_2}) \\ K_i(\bar{F} \otimes G_1) \times K_{i+1}(\bar{F} \otimes (G_1/G_2)) & \xrightarrow{\Psi_{\bar{F},i}} & K_i(\bar{F} \otimes G_2) \end{array}$$

is commutative. For $(a, b) \in K_i(F \otimes G_1) \times K_{i+1}(F \otimes (G_1/G_2))$, by the above,

$$\begin{aligned}
 K_i(\phi \otimes id_{G_2}) \Psi_{F,i}(a, b) &= K_i(\phi \otimes id_{G_2})(\Phi'_i a + \delta_{1,2,(i+1)} b) = \\
 &= K_i(\phi \otimes id_{G_2}) \Phi'_i a + K_i(\phi \otimes id_{G_2}) \delta_{1,2,(i+1)} b = \\
 &= \bar{\Phi}'_i K_i(\phi \otimes id_{G_1}) a + \bar{\delta}_{1,2,(i+1)} K_{i+1}(\phi \otimes id_{(G_1/G_2)}) b = \\
 &= \Psi_{\bar{F},i}(K_i(\phi \otimes id_{G_1}) a, K_{i+1}(\phi \otimes id_{(G_1/G_2)}) b) = \\
 &= \Psi_{\bar{F},i}(K_i(\phi \otimes id_{G_1}) \times K_{i+1}(\phi \otimes id_{(G_1/G_2)}))(a, b).
 \end{aligned}$$

Thus the above diagram is commutative. Since $G_1, G_1/G_2 \in \Upsilon$, It follows that the diagram

$$\begin{array}{ccc}
 K_i(F)^{\bar{p}(G_2)} \times K_{i+1}(F)^{\bar{q}(G_2)} & \xrightarrow{\bar{\Phi}_{i,G_2,F}} & K_i(F \otimes G_2) \\
 \downarrow K_i(\phi)^{p(G_2)} \times K_{i+1}(\phi)^{q(G_2)} & & \downarrow K_i(\phi \otimes id_{G_2}) \\
 K_i(\bar{F})^{\bar{p}(G_2)} \times K_{i+1}(\bar{F})^{\bar{q}(G_2)} & \xrightarrow{\bar{\Phi}_{i,G_2,\bar{F}}} & K_i(\bar{F} \otimes G_2)
 \end{array}$$

is commutative. Hence

$$G_2 \in \Upsilon, \quad p(G_2) = p(G_1) + q(G_1/G_2), \quad q(G_2) = q(G_1) + p(G_1/G_2),$$

$$\Phi_{i,G_2,F} = \Psi_{F,i} \circ (\Phi_{i,G_1,F} \times \Phi_{(i+1),(G_1/G_2),F}). \quad \blacksquare$$

PROPOSITION 1.5.15 *Let*

$$0 \longrightarrow G \xrightarrow{\varphi} H \xrightarrow{\psi} \mathbf{C} \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_{\mathbf{C}}$ with G nuclear and H Υ -null and let δ_i^F denote the index maps associated to the exact sequence in \mathfrak{M}_E

$$0 \longrightarrow F \otimes G \xrightarrow{id_F \otimes \varphi} F \otimes H \xrightarrow{id_F \otimes \psi} F \longrightarrow 0.$$

Then

$$G \in \Upsilon, \quad p(G) = 0, \quad q(G) = 1, \quad \Phi_{i,G,F} = \delta_{i+1}^F,$$

$$(0 \longrightarrow F \otimes G \xrightarrow{id_F \otimes \varphi} F \otimes H \xrightarrow{id_F \otimes \psi} F \longrightarrow 0) \in G_{\Upsilon}.$$

By Proposition 1.5.6 b) and Proposition 1.5.4 b),

$$G \in \Upsilon, \quad p(G) = 0, \quad q(G) = 1, \quad \Phi_{i,G,F} = \delta_{i+1}^F.$$

Since the diagram

$$\begin{array}{ccc} K_{i+1}(F) & \xrightarrow{\delta_{i+1}^F} & K_i(F \otimes G) \\ \Phi_{i,G,F} = \delta_{i+1}^F \downarrow & & \downarrow \Phi_{(i+1),G,(F \otimes G)} = \delta_{G,i}^F \\ K_i(F \otimes G) & \xrightarrow{\delta_{G,i}^F} & K_{i+1}((F \otimes G) \otimes G) \end{array}$$

is obviously commutative,

$$(0 \longrightarrow F \otimes G \xrightarrow{id_F \otimes \varphi} F \otimes H \xrightarrow{id_F \otimes \psi} F \longrightarrow 0) \in G_\Upsilon.$$

■

1.6 The Class Υ_1

Throughout this section F denotes an E - C^* -algebra.

DEFINITION 1.6.1 We denote by Υ_1 the class of unital C^* -algebras G belonging to Υ such that

$$p(G) = 1, \quad q(G) = 0, \quad \Phi_{i,G,F} = K_i(\phi_{G,F}),$$

where

$$\phi_{G,F} : F \longrightarrow F \otimes G, \quad x \longmapsto x \otimes 1_G.$$

PROPOSITION 1.6.2 $\mathbb{C} \in \Upsilon_1$.

In fact

$$\phi_{\mathbb{C},F} : F \longrightarrow F \otimes \mathbb{C}, \quad x \longmapsto x \otimes 1_{\mathbb{C}}$$

is an isomorphism.

■

PROPOSITION 1.6.3 Let $G \in \Upsilon_1$ and let $F \xrightarrow{\phi} F'$ be a morphism in \mathfrak{M}_E . If we identify $K_i(F)$ with $K_i(F \otimes G)$ for all E - C^* -algebras F using the group isomorphisms $\Phi_{i,G,F}$ then $K_i(\phi \otimes id_G)$ is identified with $K_i(\phi)$.

The assertion follows from the commutativity of the diagram

$$\begin{array}{ccc} K_i(F) & \xrightarrow{K_i(\phi)} & K_i(F') \\ \Phi_{i,G,F} \downarrow & & \downarrow \Phi_{i,G,F'} \\ K_i(F \otimes G) & \xrightarrow{K_i(\phi \otimes id_G)} & K_i(F' \otimes G) \end{array}$$

■

PROPOSITION 1.6.4 *Let G, H be C^* -algebras and $\phi : G \longrightarrow H$ and $\psi : H \longrightarrow G$ be a homotopy such that ϕ and ψ are unital. If $G \in \Upsilon_1$ then $H \in \Upsilon_1$.*

By Proposition 1.5.4 c),

$$H \in \Upsilon, \quad p(H) = 1, \quad q(H) = 0,$$

$$\Phi_{i,H,F} = K_i(id_F \otimes \phi) \circ \Phi_{i,G,F} = K_i(id_F \otimes \phi) \circ K_i(\phi_{G,F}) = K_i(\phi_{H,F}).$$

■

PROPOSITION 1.6.5 *If $(G_j)_{j \in J}$ is a finite family in Υ_1 , $J \neq \emptyset$, then $\bigotimes_{j \in J} G_j \in \Upsilon_1$.*

$\bigotimes_{j \in J} G_j$ is unital and by Proposition 1.5.12 a), $\bigotimes_{j \in J} G_j \in \Upsilon$. Assume $J = \{1, 2\}$ and let $F \xrightarrow{\phi} F'$ be a morphism in \mathfrak{M}_E . Then the diagram

$$\begin{array}{ccccc} K_i(F) & \xrightarrow{K_i(\phi_{G_1,F})} & K_i(F \otimes G_1) & \xrightarrow{K_i(\phi_{G_2,(F \otimes G_1)})} & K_i(F \otimes G_1 \otimes G_2) \\ K_i(\phi) \downarrow & & \downarrow K_i(\phi \otimes id_{G_1}) & & \downarrow K_i(\phi \otimes id_{(G_1 \otimes G_2)}) \\ K_i(F') & \xrightarrow{K_i(\phi_{G_1,F'})} & K_i(F' \otimes G_1) & \xrightarrow{K_i(\phi_{G_2,(F' \otimes G_1)})} & K_i(F' \otimes G_1 \otimes G_2) \end{array}$$

is commutative. Since

$$\phi_{(G_1 \otimes G_2),F} = \phi_{G_2,(F \otimes G_1)} \circ \phi_{G_1,F}, \quad \phi_{(G_1 \otimes G_2),F'} = \phi_{G_2,(F' \otimes G_1)} \circ \phi_{G_1,F'},$$

the diagram

$$\begin{array}{ccc} K_i(F) & \xrightarrow{K_i(\phi_{(G_1 \otimes G_2),F})} & K_i(F \otimes G_1 \otimes G_2) \\ K_i(\phi) \downarrow & & \downarrow K_i(\phi \otimes id_{(G_1 \otimes G_2)}) \\ K_i(F') & \xrightarrow{K_i(\phi_{(G_1 \otimes G_2),F'})} & K_i(F' \otimes G_1 \otimes G_2) \end{array}$$

is commutative, which proves the assertion in this case. The general case follows now by induction with respect to $\text{Card } J$. ■

PROPOSITION 1.6.6 *If $G \in \Upsilon_1$ is nuclear then every exact sequence in \mathfrak{M}_E belongs to G_Υ .*

Let

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

be an exact sequence in \mathfrak{M}_E . Then the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1 & \xrightarrow{\phi_1} & F_2 & \xrightarrow{\phi_2} & F_3 & \longrightarrow & 0 \\ & & \phi_{G,F_1} \downarrow & & \phi_{G,F_2} \downarrow & & \downarrow \phi_{G,F_3} & & \\ 0 & \longrightarrow & F_1 \otimes G & \xrightarrow{\phi_1 \otimes id_G} & F_2 \otimes G & \xrightarrow{\phi_2 \otimes id_G} & F_3 \otimes G & \longrightarrow & 0 \end{array}$$

is commutative and has exact rows. By the commutativity of the index maps (Axiom 1.2.8) the diagram

$$\begin{array}{ccc} K_i(F_3) & \xrightarrow{\delta_i} & K_{i+1}(F_1) \\ \Phi_{i,G,F_3} = K_i(\phi_{G,F_3}) \downarrow & & \downarrow \Phi_{(i+1),G,F_1} = K_{i+1}(\phi_{G,F_1}) \\ K_i(F_3 \otimes G) & \xrightarrow{\delta_{G,i}} & K_{i+1}(F_1 \otimes G) \end{array}$$

is commutative, where δ_i denotes the index maps of the exact sequence

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0.$$
■

PROPOSITION 1.6.7 *Let G be a C^* -algebra.*

- a) $\phi_{\tilde{G},F} = (id_F \otimes \lambda_G) \circ \phi_{\mathbb{C},F}$.
- b) G is Υ -null iff $\tilde{G} \in \Upsilon_1$.
- c) If G is Υ -null and $\phi : G \longrightarrow G'$, $\psi : G \longrightarrow G'$ are C^* -homomorphisms then $K_i(id_F \otimes \tilde{\phi}) = K_i(id_F \otimes \tilde{\psi})$. In particular if $G = G'$ then

$$K_i(id_F \otimes \tilde{\phi}) = id_{K_i(F \otimes \tilde{G})} \approx id_{K_i(F)}.$$

a) is easy to see.

b) By Corollary 1.4.5 b), the sequence

$$0 \longrightarrow K_i(F \otimes G) \xrightarrow{K_i(id_F \otimes \iota_G)} K_i(F \otimes \tilde{G}) \xrightleftharpoons[K_i(id_F \otimes \lambda_G)]{K_i(id_F \otimes \pi_G)} K_i(F) \longrightarrow 0$$

is split exact. By a) and Proposition 1.6.2,

$$K_i(\phi_{\tilde{G},F}) = K_i(id_F \otimes \lambda_G) \circ \Phi_{i,\mathbf{C},F}.$$

If $\tilde{G} \in \Upsilon_1$ then

$$\Phi_{i,\tilde{G},F} = K_i(\phi_{\tilde{G},F}) = K_i(id_F \otimes \lambda_G) \circ \Phi_{i,\mathbf{C},F},$$

so by Proposition 1.6.2, $K_i(id_F \otimes \lambda_G)$ is an isomorphism, $K_i(id_F \otimes \iota_G) = 0$, $K_i(F \otimes G) = 0$, and G is Υ -null. If G is Υ -null then $K_i(id_F \otimes \lambda_G)$ is an isomorphism so

$$K_i(\phi_{\tilde{G},F}) : K_i(F) \longrightarrow K_i(F \otimes G)$$

is an isomorphism and $\tilde{G} \in \Upsilon_1$.

c) Since $\tilde{\varphi} \circ \lambda_G = \tilde{\psi} \circ \lambda_G$,

$$K_i(id_F \otimes \tilde{\varphi}) \circ K_i(id_F \otimes \lambda_G) = K_i(id_F \otimes \tilde{\psi}) \circ K_i(id_F \otimes \lambda_G).$$

By b), $\tilde{G} \in \Upsilon_1$ and so $K_i(id_F \otimes \lambda_G)$ is an isomorphism. Thus $K_i(id_F \otimes \tilde{\varphi}) = K_i(id_F \otimes \tilde{\psi})$. ■

COROLLARY 1.6.8 *If $(G_j)_{j \in J}$ is a finite family of Υ -null C^* -algebras and $G := \prod_{j \in J} G_j$ then $\tilde{G} \in \Upsilon_1$.*

By Proposition 1.5.9 a), G is Υ -null and by Proposition 1.6.7 b), $\tilde{G} \in \Upsilon_1$. ■

PROPOSITION 1.6.9 *Let*

$$0 \longrightarrow G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\psi} G_3 \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_{\mathbf{C}}$ such that G_1 is Υ -null, G_3 is nuclear, and G_2, G_3 are unital. Then $G_2 \in \Upsilon_1$ iff $G_3 \in \Upsilon_1$.

Since G_2 and G_3 are unital and ψ is surjective, $\psi(1_{G_2}) = 1_{G_3}$. It follows

$$\phi_{G_3,F} = (id_F \otimes \psi) \circ \phi_{G_2,F}, \quad K_i(\phi_{G_3,F}) = K_i(id_F \otimes \psi) \circ K_i(\phi_{G_2,F}).$$

By Proposition 1.5.6 a), $K_i(id_F \otimes \psi)$ is a group isomorphism,

$$G_2, G_3 \in \Upsilon, \quad p(G_2) = p(G_3) = 1, \quad q(G_2) = q(G_3) = 0,$$

$$\Phi_{i,G_3,F} = K_i(id_F \otimes \psi) \circ \Phi_{i,G_2,F}.$$

If $G_2 \in \Upsilon_1$ then by the above,

$$\Phi_{i,G_3,F} = K_i(id_F \otimes \psi) \circ K_i(\phi_{G_2,F}) = K_i(\phi_{G_3,F}),$$

so $G_3 \in \Upsilon_1$. If $G_3 \in \Upsilon_1$ then by the above,

$$K_i(id_F \otimes \psi) \circ K_i(\phi_{G_2,F}) = K_i(\phi_{G_3,F}) = \Phi_{i,G_3,F} = K_i(id_F \otimes \psi) \circ \Phi_{i,G_2,F},$$

so $\Phi_{i,G_2,F} = K_i(\phi_{G_2,F})$ and $G_2 \in \Upsilon_1$. ■

Chapter 2

Locally Compact Spaces

2.1 Tietze's Theorem

DEFINITION 2.1.1 Let Ω be a topological space and F an E - C^* -algebra. We endow canonically the C^* -algebra $\mathcal{C}(\Omega, F)$ with the structure of an E - C^* -algebra by putting

$$\alpha x : \Omega \longrightarrow F, \quad \omega \longmapsto \alpha x(\omega)$$

for all $(\alpha, x) \in E \times F$. If Ω is a locally compact space then we endow $\mathcal{C}_0(\Omega, F)$ with the structure of an E - C^* -algebra in a similar way. If Ω' is an open set of a locally compact space Ω then we identify $\mathcal{C}_0(\Omega', F)$ with the E -ideal $\{x \in \mathcal{C}_0(\Omega, F) \mid x|(\Omega \setminus \Omega') = 0\}$ of $\mathcal{C}_0(\Omega, F)$.

DEFINITION 2.1.2 Let Ω be a locally compact space with $\mathcal{C}_0(\Omega, \mathbb{C}) \in \Upsilon$. We put

$$\Omega \in \Upsilon, \quad p(\Omega) := p(\mathcal{C}_0(\Omega, \mathbb{C})), \quad q(\Omega) := q(\mathcal{C}_0(\Omega, \mathbb{C})),$$

$$\Phi_{i, \Omega, F} := \Phi_{i, \mathcal{C}_0(\Omega, \mathbb{C}), F}, \quad \Omega_\Upsilon := \mathcal{C}_0(\Omega, \mathbb{C})_\Upsilon, \quad \Omega \in \Upsilon_1 : \Longleftrightarrow \mathcal{C}_0(\Omega, \mathbb{C}) \in \Upsilon_1.$$

We say that Ω is Υ -null if $\mathcal{C}_0(\Omega, \mathbb{C})$ is Υ -null. We say that Ω is **null-homotopic** if $\mathcal{C}_0(\Omega, \mathbb{C})$ is null-homotopic.

PROPOSITION 2.1.3 If Ω is a locally compact space and if Ω^* denotes its Alexandroff compactification then Ω is Υ -null iff $\Omega^* \in \Upsilon_1$.

The Proposition is a particular case of Proposition 1.6.7. ■

LEMMA 2.1.4 Let Ω be a locally compact space.

- a) $\mathcal{C}_0(\Omega, \mathbb{C})$ is nuclear.
- b) $\mathcal{C}_0(\Omega, F) \approx F \otimes \mathcal{C}_0(\Omega, \mathbb{C})$.
- c) If Ω is a finite compact space then $\Omega \in \Upsilon$, $p(\Omega) = \text{Card } \Omega$, $q(\Omega) = 0$, and every exact sequence in \mathfrak{M}_E belongs to Ω_Υ .

a) [5] Theorem T.6.20.

b) [5] Proposition T.5.11,

c) follows from Proposition 1.5.9 c). ■

COROLLARY 2.1.5 (Tietze's Theorem) *Let Ω be a locally compact space, Γ a closed set of Ω , $\varphi : \mathcal{C}_0(\Omega \setminus \Gamma, F) \longrightarrow \mathcal{C}_0(\Omega, F)$ the inclusion map, and*

$$\psi : \mathcal{C}_0(\Omega, F) \longrightarrow \mathcal{C}_0(\Gamma, F), \quad x \longmapsto x|_{\Gamma}.$$

Then

$$0 \longrightarrow \mathcal{C}_0(\Omega \setminus \Gamma, F) \xrightarrow{\varphi} \mathcal{C}_0(\Omega, F) \xrightarrow{\psi} \mathcal{C}_0(\Gamma, F) \longrightarrow 0$$

is an exact sequence in \mathfrak{M}_E .

By Lemma 2.1.4 a),b), the assertion follows from Proposition 1.4.9. ■

COROLLARY 2.1.6 *If*

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

is an exact sequence in \mathfrak{M}_E and Ω a locally compact space then

$$0 \longrightarrow \mathcal{C}_0(\Omega, F_1) \xrightarrow{\phi_1 \otimes id_G} \mathcal{C}_0(\Omega, F_2) \xrightarrow{\phi_2 \otimes id_G} \mathcal{C}_0(\Omega, F_3) \longrightarrow 0$$

is an exact sequence in \mathfrak{M}_E .

By Lemma 2.1.4 a),b), the assertion follows from Proposition 1.4.10. ■

PROPOSITION 2.1.7 *Let*

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

be an exact sequence in \mathfrak{M}_E , Ω a locally compact space, Γ a closed set of Ω , $\varphi : \mathcal{C}_0(\Omega \setminus \Gamma, \mathbb{C}) \longrightarrow \mathcal{C}_0(\Omega, \mathbb{C})$ the inclusion map, and

$$\psi : \mathcal{C}_0(\Omega, \mathbb{C}) \longrightarrow \mathcal{C}_0(\Gamma, \mathbb{C}), \quad x \longmapsto x|_{\Gamma}.$$

a) $G := \{x \in \mathcal{C}_0(\Omega, F_2) \mid x|_{\Gamma} \in \mathcal{C}_0(\Gamma, F_1)\}$ is a closed E -ideal of $\mathcal{C}_0(\Omega, F_2)$; we denote by $\varphi' : G \longrightarrow \mathcal{C}_0(\Omega, F_2)$ the inclusion map.

b) The sequence in \mathfrak{M}_E

$$0 \longrightarrow G \xrightarrow{\varphi'} \mathcal{C}_0(\Omega, F_2) \xrightarrow{\phi_2 \otimes \psi} \mathcal{C}_0(\Gamma, F_3) \longrightarrow 0$$

is exact.

a) is easy to see.

b) We put

$$G_1 := \mathcal{C}_0(\Omega \setminus \Gamma, \mathbf{C}), \quad G_2 := \mathcal{C}_0(\Omega, \mathbf{C}), \quad G_3 := \mathcal{C}_0(\Gamma, \mathbf{C}).$$

Let us consider the following commutative diagram.

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & F_1 \otimes G_1 & \xrightarrow{\phi_1 \otimes id_{G_1}} & F_2 \otimes G_1 & \xrightarrow{\phi_2 \otimes id_{G_1}} & F_3 \otimes G_1 \longrightarrow 0 \\
 & & id_{F_1} \otimes \varphi \downarrow & & id_{F_2} \otimes \varphi \downarrow & & id_{F_3} \otimes \varphi \downarrow \\
 0 & \longrightarrow & F_1 \otimes G_2 & \xrightarrow{\phi_1 \otimes id_{G_2}} & F_2 \otimes G_2 & \xrightarrow{\phi_2 \otimes id_{G_2}} & F_3 \otimes G_2 \longrightarrow 0 \\
 & & id_{F_1} \otimes \psi \downarrow & & id_{F_2} \otimes \psi \downarrow & & id_{F_3} \otimes \psi \downarrow \\
 0 & \longrightarrow & F_1 \otimes G_3 & \xrightarrow{\phi_1 \otimes id_{G_3}} & F_2 \otimes G_3 & \xrightarrow{\phi_2 \otimes id_{G_3}} & F_3 \otimes G_3 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

By Lemma 2.1.4 a), Proposition 1.4.9, and Proposition 1.4.10, its columns and rows are exact. It follows that $\phi_2 \otimes \psi$ is surjective. Let $x \in \text{Ker}(\phi_2 \otimes \psi)$. Then

$$(id_{F_3} \otimes \psi)(\phi_2 \otimes id_{G_2})x = (\phi_2 \otimes \psi)x = 0,$$

so there is a $y \in F_2 \otimes G_1$ with

$$(\phi_2 \otimes \varphi)y = (id_{F_3} \otimes \varphi)(\phi_2 \otimes id_{G_1})y = (\phi_2 \otimes id_{G_2})x.$$

Then

$$(\phi_2 \otimes id_{G_2})(x - (id_{F_2} \otimes \phi)y) = (\phi_2 \otimes id_{G_2})x - (\phi_2 \otimes \phi)y = 0,$$

so there is a $z \in F_1 \otimes G_2$ with

$$(\phi_1 \otimes id_{G_2})z = x - (id_{F_2} \otimes \phi)y.$$

Thus

$$x = (id_{F_2} \otimes \phi)y + (\phi_1 \otimes id_{G_2})z \in G, \quad Ker(\phi_2 \otimes \psi) \subset G.$$

Let now $x \in G$. By Proposition 1.4.9, there is a $y \in \mathcal{C}_0(\Omega, F_1) = F_1 \otimes G_2$ with $x|_\Gamma = y|_\Gamma$. There is a $z \in \mathcal{C}_0(\Omega \setminus \Gamma, F_2) = F_2 \otimes G_1$ with

$$(id_{F_2} \otimes \phi)z = x - (\phi_1 \otimes id_{G_2})y.$$

We get

$$\begin{aligned} (\phi_2 \otimes \psi)x &= (\phi_2 \otimes \psi)(\phi_1 \otimes id_{G_2})y + (\phi_2 \otimes \psi)(id_{F_2} \otimes \phi)z = \\ &= ((\phi_2 \circ \phi_1) \otimes \psi)y + (\phi_2 \otimes (\psi \circ \phi))z = 0, \end{aligned}$$

$$G \subset Ker(\phi_2 \otimes \psi). \quad \blacksquare$$

Remark. If we put $F_1 := 0$ and $F_2 = F_3$ in the above Proposition then we obtain Tietze's Theorem (Corollary 2.1.5).

PROPOSITION 2.1.8 (Topological six-term sequence) *Let Ω be a locally compact space, Γ a closed set of Ω , $\phi : \mathcal{C}_0(\Omega \setminus \Gamma, F) \longrightarrow \mathcal{C}_0(\Omega, F)$ the inclusion map,*

$$\psi : \mathcal{C}_0(\Omega, F) \longrightarrow \mathcal{C}_0(\Gamma, F), \quad x \longmapsto x|_\Gamma,$$

and δ_i the index maps associated to the exact sequence in \mathfrak{M}_E (Tietze's Theorem (Corollary 2.1.5))

$$0 \longrightarrow \mathcal{C}_0(\Omega \setminus \Gamma, F) \xrightarrow{\phi} \mathcal{C}_0(\Omega, F) \xrightarrow{\psi} \mathcal{C}_0(\Gamma, F) \longrightarrow 0.$$

a) Assume $\Omega \setminus \Gamma$ is Υ -null.

$a_1)$ $K_i(\psi) : K_i(\mathcal{C}_0(\Omega, F)) \longrightarrow K_i(\mathcal{C}_0(\Gamma, F))$ is a group isomorphism.

$a_2)$ If $\Omega \in \Upsilon$ or $\Gamma \in \Upsilon$ then

$$\Omega, \Gamma \in \Upsilon, \quad p(\Omega) = p(\Gamma), \quad q(\Omega) = q(\Gamma),$$

$$\Phi_{i,\Gamma,F} = K_i(id_F \otimes \Psi) \circ \Phi_{i,\Omega,F}, \quad \Omega_\Upsilon = \Gamma_\Upsilon.$$

$b)$ Assume Ω is Υ -null.

$b_1)$ $\delta_i : K_i(\mathcal{C}_0(\Gamma, F)) \longrightarrow K_{i+1}(\mathcal{C}_0(\Omega \setminus \Gamma, F))$ is a group isomorphism.

$b_2)$ If $\Omega \setminus \Gamma \in \Upsilon$ or $\Gamma \in \Upsilon$ then

$$\Omega \setminus \Gamma, \Gamma \in \Upsilon, \quad p(\Omega \setminus \Gamma) = q(\Gamma), \quad q(\Omega \setminus \Gamma) = p(\Gamma),$$

$$\Phi_{i,\Gamma,F} = \Phi_{(i+1),(\Omega \setminus \Gamma),F} \circ \delta_i.$$

$c)$ Assume Γ is Υ -null.

$c_1)$ $K_i(\varphi) : K_i(\mathcal{C}_0(\Omega \setminus \Gamma, F)) \longrightarrow K_i(\mathcal{C}_0(\Omega, F))$ is a group isomorphism.

$c_2)$ If $\Omega \setminus \Gamma \in \Upsilon$ or $\Omega \in \Upsilon$ then

$$\Omega \setminus \Gamma, \Omega \in \Upsilon, \quad p(\Omega \setminus \Gamma) = p(\Omega), \quad q(\Omega \setminus \Gamma) = q(\Omega),$$

$$\Phi_{i,\Omega,F} = K_i(id_F \otimes \varphi) \circ \Phi_{i,(\Omega \setminus \Gamma),F}, \quad (\Omega \setminus \Gamma)_\Upsilon = \Omega_\Upsilon.$$

The assertions follow from Lemma 2.1.4 a),b) and Proposition 1.5.6. ■

COROLLARY 2.1.9 Let Ω be a locally compact space, $\omega \in \Omega$ such that $\Omega \setminus \{\omega\}$ is Υ -null, Γ a closed set of Ω ,

$$\Omega' := (\Omega \setminus \{\omega\}) \setminus \Gamma, \quad \Gamma' := \Gamma \setminus \{\omega\},$$

$\varphi : \mathcal{C}_0(\Omega', F) \longrightarrow \mathcal{C}_0(\Omega \setminus \{\omega\}, F)$ the inclusion map,

$$\psi : \mathcal{C}_0(\Omega \setminus \{\omega\}, F) \longrightarrow \mathcal{C}_0(\Gamma', F), \quad x \longmapsto x|_{\Gamma'},$$

and δ_i the index maps of the exact sequence in \mathfrak{M}_E (Tietze's Theorem (Corollary 2.1.5))

$$0 \longrightarrow \mathcal{C}_0(\Omega', F) \xrightarrow{\varphi} \mathcal{C}_0(\Omega \setminus \{\omega\}, F) \xrightarrow{\psi} \mathcal{C}_0(\Gamma', F) \longrightarrow 0.$$

$a)$ $\delta_i : K_i(\mathcal{C}_0(\Gamma', F)) \longrightarrow K_{i+1}(\mathcal{C}_0(\Omega', F))$ is a group isomorphism.

b) If $\Omega' \in \Upsilon$ or $\Gamma' \in \Upsilon$ then

$$\Omega', \Gamma' \in \Upsilon, \quad p(\Omega') = q(\Gamma'), \quad q(\Omega') = p(\Gamma'),$$

$$\Phi_{i, \Gamma', F} = \Phi_{(i+1), \Omega', F} \circ \delta_i.$$

c) If Γ is finite then

$$\Omega' \in \Upsilon, \quad p(\Omega') = 0, \quad q(\Omega') = \text{Card } \Gamma'.$$

a) and b) follow from the Topological six-term sequence (Proposition 2.1.8 b)).

c) follows from b) and Lemma 2.1.4 c). ■

COROLLARY 2.1.10 *Let Ω, Ω' be locally compact spaces, $\omega \in \Omega$, and $\omega' \in \Omega'$ such that $\Omega' \setminus \{\omega'\}$ is null-homotopic.*

$$a) \quad K_i(\mathcal{C}_0((\Omega \times \Omega') \setminus \{(\omega, \omega')\}, F)) \approx K_i(\mathcal{C}_0((\Omega \setminus \{\omega\}) \times \Omega', F)).$$

b) *If also $\Omega \setminus \{\omega\}$ is null-homotopic then $\mathcal{C}_0((\Omega \times \Omega') \setminus \{(\omega, \omega')\}, F)$ is K-null.*

a) The sequence in \mathfrak{M}_E (with obvious notation)

$$\begin{aligned} 0 \longrightarrow \mathcal{C}_0((\Omega \setminus \{\omega\}) \times \Omega', F) &\xrightarrow{\varphi} \mathcal{C}_0((\Omega \times \Omega') \setminus \{(\omega, \omega')\}, F) \\ \mathcal{C}_0((\Omega \times \Omega') \setminus \{(\omega, \omega')\}, F) &\xrightarrow{\psi} \mathcal{C}_0(\{\omega\} \times (\Omega' \setminus \{\omega'\}), F) \longrightarrow 0 \end{aligned}$$

is exact and the assertion follows from the Topological six-term sequence (Proposition 2.1.8 c₁)).

b) By Proposition 1.4.2 c) and Lemma 2.1.4 b), $(\Omega \setminus \{\omega\}) \times \Omega'$ is null-homotopic and so K-null (Proposition 1.5.4 a)). By a),

$$K_i(\mathcal{C}_0((\Omega \times \Omega') \setminus \{(\omega, \omega')\}, F))$$

is K-null. ■

PROPOSITION 2.1.11 (Topological triple) *Let Ω_1 be a locally compact space, Ω_2 an open set of Ω_1 , Ω_3 an open set of Ω_2 , and $\varphi : \mathcal{C}_0(\Omega_2 \setminus \Omega_3, F) \longrightarrow \mathcal{C}_0(\Omega_1 \setminus \Omega_3, F)$ the inclusion map. For all $j, k \in \{1, 2, 3\}$, $j < k$, put*

$$\psi_{j,k} : \mathcal{C}_0(\Omega_j, F) \longrightarrow \mathcal{C}_0(\Omega_j \setminus \Omega_k, F), \quad x \longmapsto x|(\Omega_j \setminus \Omega_k)$$

and denote by $\varphi_{j,k} : \mathcal{C}_0(\Omega_k, F) \longrightarrow \mathcal{C}_0(\Omega_j, F)$ the inclusion map and by $\delta_{j,k,i}$ the index maps associated to the exact sequence in \mathfrak{M}_E

$$0 \longrightarrow \mathcal{C}_0(\Omega_k, F) \xrightarrow{\varphi_{j,k}} \mathcal{C}_0(\Omega_j, F) \xrightarrow{\psi_{j,k}} \mathcal{C}_0(\Omega_j \setminus \Omega_k, F) \longrightarrow 0.$$

a) Assume $\mathcal{C}_0(\Omega_2, F)$ K -null.

$a_1)$ $\delta_{2,3,i} : K_i(\mathcal{C}_0(\Omega_2 \setminus \Omega_3, F)) \longrightarrow K_{i+1}(\mathcal{C}_0(\Omega_3, F))$ is a group isomorphism.

$a_2)$ $\delta_{2,3,i} = \delta_{1,3,i} \circ K_i(\varphi)$.

$a_3)$ $\varphi_{1,3}$ is K -null.

$a_4)$ If we put $\Phi_i := K_i(\varphi) \circ (\delta_{2,3,i})^{-1}$ then

$$\begin{array}{ccc} 0 \longrightarrow K_i(\mathcal{C}_0(\Omega_1, F)) & \xrightarrow{K_i(\psi_{1,3})} & K_i(\mathcal{C}_0(\Omega_1 \setminus \Omega_3, F)) \xleftarrow{\delta_{1,3,i}} \\ & & \xleftarrow{\Phi_i} K_{i+1}(\mathcal{C}_0(\Omega_3, F)) \longrightarrow 0 \end{array}$$

is a split exact sequence and the map

$$K_i(\mathcal{C}_0(\Omega_1, F)) \times K_{i+1}(\mathcal{C}_0(\Omega_3, F)) \longrightarrow K_i(\mathcal{C}_0(\Omega_1 \setminus \Omega_3, F)),$$

$$(a, b) \longmapsto K_i(\psi_{1,3})a + \Phi_i b$$

is a group isomorphism.

$a_5)$ If Ω_2 is Υ -null and $\Omega_1, \Omega_3 \in \Upsilon$ then

$$\Omega_1 \setminus \Omega_3 \in \Upsilon, \quad p(\Omega_1 \setminus \Omega_3) = p(\Omega_1) + q(\Omega_3), \quad q(\Omega_1 \setminus \Omega_3) = q(\Omega_1) + p(\Omega_3),$$

and (with the notation of Proposition 1.5.13)

$$\Phi_{i,(\Omega_1 \setminus \Omega_3),F} = \Psi_{F,i} \circ (\Phi_{i,\Omega_1,F} \times \Phi_{(i+1),\Omega_3,F}).$$

b) Assume $\mathcal{C}_0(\Omega_1 \setminus \Omega_3, F)$ K -null.

$$b_1) \quad \delta_{2,3,i} = 0.$$

$$b_2) \quad K_i(\varphi_{1,3}) : K_i(\mathcal{C}_0(\Omega_3, F)) \longrightarrow K_i(\mathcal{C}_0(\Omega_1, F)) \text{ is a group isomorphism.}$$

$$b_3) \quad \text{If we put } \Phi_i := K_i(\varphi_{1,3})^{-1} \circ K_i(\varphi_{1,2}) \text{ then the map}$$

$$\begin{aligned} \Psi : K_i(\mathcal{C}_0(\Omega_2, F)) &\longrightarrow K_i(\mathcal{C}_0(\Omega_3, F)) \times K_i(\mathcal{C}_0(\Omega_2 \setminus \Omega_3, F)), \\ b &\longmapsto (\Phi_i b, K_i(\psi_{2,3})b) \end{aligned}$$

is a group isomorphism.

$$b_4) \quad \text{If } \psi_{1,2} \text{ is } K\text{-null and if we put } \Phi'_i := K_i(\varphi_{2,3}) \circ K_i(\varphi_{1,3})^{-1} \text{ (by } c_2)) \text{ then}$$

$$\begin{aligned} 0 \longrightarrow K_{i+1}(\mathcal{C}_0(\Omega_1 \setminus \Omega_2, F)) &\xrightarrow{\delta_{1,2,(i+1)}} K_i(\mathcal{C}_0(\Omega_2, F)) \xleftarrow{\Phi'_i} \\ &\xleftarrow{\Phi'_i} K_i(\mathcal{C}_0(\Omega_1, F)) \xrightarrow{\Phi'_i} 0 \end{aligned}$$

is a split exact sequence and the map

$$\begin{aligned} K_i(\mathcal{C}_0(\Omega_1, F)) \times K_{i+1}(\mathcal{C}_0(\Omega_1 \setminus \Omega_2, F)) &\longrightarrow K_i(\mathcal{C}_0(\Omega_2, F)), \\ (a, b) &\longmapsto \Phi'_i a + \delta_{1,2,(i+1)} b \end{aligned}$$

is a group isomorphism.

$$b_5) \quad \text{If } \Omega_1 \setminus \Omega_3 \text{ is } \Upsilon\text{-null, } \Omega_1, \Omega_1 \setminus \Omega_2 \in \Upsilon, \text{ and } \psi_{1,2} \text{ is } K\text{-null then}$$

$$\Omega_2 \in \Upsilon, \quad p(\Omega_2) = p(\Omega_1) + q(\Omega_1 \setminus \Omega_2), \quad q(\Omega_2) = q(\Omega_1) + p(\Omega_1 \setminus \Omega_2).$$

$$c) \quad \text{Assume } \mathcal{C}_0(\Omega_1, F) \text{ } K\text{-null and put}$$

$$\psi : \mathcal{C}_0(\Omega_1 \setminus \Omega_3, F) \longrightarrow \mathcal{C}_0(\Omega_1 \setminus \Omega_2, F), \quad x \longmapsto x|(\Omega_1 \setminus \Omega_2).$$

$$c_1) \quad \delta_{1,2,i} \text{ and } \delta_{1,3,i} \text{ are group isomorphisms.}$$

$$c_2) \quad K_i(\varphi_{2,3}) \circ \delta_{1,3,(i+1)} = \delta_{1,2,(i+1)} \circ K_{i+1}(\psi).$$

$$c_3) \quad \text{Let } \varphi' : \mathcal{C}_0(\Omega_1 \setminus \Omega_2, F) \longrightarrow \mathcal{C}_0(\Omega_1 \setminus \Omega_3, F) \text{ be a morphism in } \mathfrak{M}_E \text{ such that}$$

$$K_i(\psi \circ \varphi') = id_{K_i(\mathcal{C}_0(\Omega_1 \setminus \Omega_2, F))}.$$

If we put

$$\Phi_i := \delta_{1,3,(i+1)} \circ K_{i+1}(\varphi') \circ (\delta_{1,2,(i+1)})^{-1}$$

then $K_i(\varphi_{2,3}) \circ \Phi_i = id_{K_i(\mathcal{C}_0(\Omega_2, F))}$. If in addition $\psi_{2,3}$ is K -null then

$$0 \longrightarrow K_{i+1}(\mathcal{C}_0(\Omega_2 \setminus \Omega_3, F)) \xrightarrow{\delta_{2,3,(i+1)}} K_i(\mathcal{C}_0(\Omega_3, F)) \xleftarrow[\Phi_i]{K_i(\varphi_{2,3})} \\ \xleftarrow[\Phi_i]{K_i(\varphi_{2,3})} K_i(\mathcal{C}_0(\Omega_2, F)) \longrightarrow 0$$

is a split exact sequence and the map

$$K_{i+1}((\mathcal{C}_0(\Omega_2 \setminus \Omega_3, F)) \times K_i(\mathcal{C}_0(\Omega_2, F)) \longrightarrow K_i(\mathcal{C}_0(\Omega_3, F)), \\ (a, b) \longmapsto \delta_{2,3,(i+1)}a + \Phi_i b$$

is a group isomorphism.

Up to a_5) and b_5) the Proposition follows from Tietze's Theorem (Corollary 2.1.5) and from the triple theorem (Theorem 1.3.8) (and Lemma 2.1.4 a),b)). a_5) follows from Proposition 1.5.13 and b_5) follows from Proposition 1.5.14. ■

COROLLARY 2.1.12 Let $F \xrightarrow{\phi} F'$ be a morphism in \mathfrak{M}_E . We use the notation and hypotheses of Proposition 2.1.11 and the hypothesis that $\mathcal{C}_0(\Omega_2, F)$ and $\mathcal{C}_0(\Omega_2, F')$ are K -null, and mark with an accent those notation associated to F' . We put for all $j \in \{1, 2, 3\}$ and for all $j, k \in \{1, 2, 3\}$, $j < k$,

$$\phi_j : \mathcal{C}_0(\Omega_j, F) \longrightarrow \mathcal{C}_0(\Omega_j, F'), \quad x \longmapsto \phi \circ x,$$

$$\phi_{j,k} : \mathcal{C}_0(\Omega_j \setminus \Omega_k, F) \longrightarrow \mathcal{C}_0(\Omega_j \setminus \Omega_k, F'), \quad x \longmapsto \phi \circ x.$$

$$a) \quad \Phi'_i \circ K_{i+1}(\phi_3) = K_i(\phi_{1,3}) \circ \Phi_i.$$

b) If we identify $K_i(\mathcal{C}_0(\Omega_1 \setminus \Omega_3, F))$ with $K_i(\mathcal{C}_0(\Omega_1, F)) \times K_{i+1}(\mathcal{C}_0(\Omega_3, F))$ and $K_i(\mathcal{C}_0(\Omega_1 \setminus \Omega_3, F'))$ with $K_i(\mathcal{C}_0(\Omega_1, F')) \times K_{i+1}(\mathcal{C}_0(\Omega_3, F'))$ using the isomorphisms of Proposition 2.1.11 a_4) then

$$K_i(\phi_{1,3}) : K_i(\mathcal{C}_0(\Omega_1 \setminus \Omega_3, F)) \longrightarrow K_i(\mathcal{C}_0(\Omega_1 \setminus \Omega_3, F')),$$

$$(a, b) \longrightarrow (K_i(\phi_1)a, K_{i+1}(\phi_3)b)$$

is a group isomorphism.

a) The diagram

$$\begin{array}{ccccccc}
 0 \longrightarrow \mathcal{C}_0(\Omega_3, F) & \xrightarrow{\phi_{2,3}} & \mathcal{C}_0(\Omega_2, F) & \xrightarrow{\psi_{2,3}} & \mathcal{C}_0(\Omega_2 \setminus \Omega_3, F) & \longrightarrow & 0 \\
 \downarrow \phi_3 & & \downarrow \phi_2 & & \downarrow \phi_{2,3} & & \\
 0 \longrightarrow \mathcal{C}_0(\Omega_3, F') & \xrightarrow{\phi'_{2,3}} & \mathcal{C}_0(\Omega_2, F') & \xrightarrow{\psi'_{2,3}} & \mathcal{C}_0(\Omega_2 \setminus \Omega_3, F') & \longrightarrow & 0
 \end{array}$$

is obviously commutative and has exact rows. By the commutativity of the index maps (Axiom 1.2.8),

$$\begin{aligned}
 K_{i+1}(\phi_3) \circ \delta_{2,3,i} &= (\delta_{2,3,i})' \circ K_i(\phi_{2,3}), \\
 ((\delta_{2,3,i})')^{-1} \circ K_{i+1}(\phi_3) &= K_i(\phi_{2,3}) \circ (\delta_{2,3,i})^{-1}.
 \end{aligned}$$

By the above, since $\phi_{1,3} \circ \varphi = \varphi' \circ \phi_{2,3}$,

$$\begin{aligned}
 K_i(\phi_{1,3}) \circ \Phi_i &= K_i(\phi_{1,3}) \circ K_i(\varphi) \circ (\delta_{2,3,i})^{-1} = K_i(\varphi') \circ K_i(\phi_{2,3}) \circ (\delta_{2,3,i})^{-1} = \\
 &= K_i(\varphi') \circ ((\delta_{2,3,i})')^{-1} \circ K_{i+1}(\phi_3) = \Phi'_i \circ K_{i+1}(\phi_3).
 \end{aligned}$$

b) follows from a) and Proposition 2.1.11 a4). ■

2.2 Alexandroff Compactification

THEOREM 2.2.1 (Alexandroff K-theorem) *Let Ω be a locally compact space and Ω^* its Alexandroff compactification. We denote by*

$$\varphi : \mathcal{C}_0(\Omega, F) \longrightarrow \mathcal{C}(\Omega^*, F)$$

the inclusion map and put

$$\lambda : F \longrightarrow \mathcal{C}(\Omega^*, F), \quad y \longmapsto y1_{\mathcal{C}(\Omega^*, \mathbb{C})}$$

a) *The map*

$$K_i(\mathcal{C}_0(\Omega, F)) \times K_i(F) \longrightarrow K_i(\mathcal{C}(\Omega^*, F)), \quad (a, b) \longmapsto K_i(\varphi)a + K_i(\lambda)b$$

is a group isomorphism.

b) If $\Omega \in \Upsilon$ then

$$\Omega^* \in \Upsilon, \quad p(\Omega^*) = p(\Omega) + 1, \quad q(\Omega^*) = q(\Omega) \quad \Omega_\Upsilon \subset \Omega_\Upsilon^*.$$

c) Ω is Υ -null iff $\Omega^* \in \Upsilon_1$.

$\mathcal{C}(\Omega^*, \mathbb{C})$ is the unitization of $\mathcal{C}_0(\Omega, \mathbb{C})$.

a) Since

$$\mathcal{C}_0(\Omega, F) \approx F \otimes \mathcal{C}_0(\Omega, \mathbb{C}), \quad \mathcal{C}_0(\Omega^*, F) \approx F \otimes \mathcal{C}_0(\Omega^*, \mathbb{C})$$

(Lemma 2.1.4 b)), the assertion follows from Corollary 1.4.5 b).

b) follows from Corollary 1.5.8.

c) follows from Proposition 1.6.7 b). ■

COROLLARY 2.2.2 *Let Ω_1 and Ω_2 be locally compact spaces, Ω_1^*, Ω_2^* their Alexandroff compactification, respectively, $\vartheta : \Omega_1 \longrightarrow \Omega_2$ a proper continuous map, $\vartheta^* : \Omega_1^* \longrightarrow \Omega_2^*$ its continuous extension, and*

$$\phi : \mathcal{C}_0(\Omega_2, F) \longrightarrow \mathcal{C}_0(\Omega_1, F), \quad x \longmapsto x \circ \vartheta,$$

$$\phi^* : \mathcal{C}(\Omega_2^*, F) \longrightarrow \mathcal{C}(\Omega_1^*, F), \quad x \longmapsto x \circ \vartheta^*.$$

a) *If we identify $K_i(\mathcal{C}(\Omega_j^*, F))$ with $K_i(\mathcal{C}_0(\Omega_j, F)) \times K_i(F)$ for every $j \in \{1, 2\}$ using the group isomorphisms of the Alexandroff K-theorem (Theorem 2.2.1 a)) then*

$$K_i(\phi^*) : K_i(\mathcal{C}(\Omega_2^*, F)) \longrightarrow K_i(\mathcal{C}(\Omega_1^*, F)), \quad (a, b) \longmapsto (K_i(\phi)a, b).$$

b) *Let $\vartheta' : \Omega_1 \longrightarrow \Omega_2$ be a proper continuous map and let ϕ', ϕ'^* be the above maps associated to ϑ' . If Ω_2 is Υ -null then $K_i(id_F \otimes \phi^*) = K_i(id_F \otimes \phi'^*)$. In particular if $\Omega_1 = \Omega_2$ then*

$$K_i(id_F \otimes \phi^*) = id_{K_i(\mathcal{C}(\Omega_1^*, F))}.$$

a) follows from Corollary 1.4.5 c).

b) follows from Proposition 1.6.7 c). ■

COROLLARY 2.2.3 *Let $F \xrightarrow{\phi} F'$ be a morphism in \mathfrak{M}_E . We use the notation of the Alexandroff K-theorem (Theorem 2.2.1) and put*

$$\phi_{\Omega} : \mathcal{C}_0(\Omega, F) \longrightarrow \mathcal{C}_0(\Omega, F'), \quad x \longmapsto \phi \circ x,$$

$$\phi_{\Omega^*} : \mathcal{C}(\Omega^*, F) \longrightarrow \mathcal{C}(\Omega^*, F'), \quad x \longmapsto \phi \circ x.$$

If we identify $K_i(\mathcal{C}(\Omega^, F))$ with $K_i(\mathcal{C}_0(\Omega, F)) \times K_i(F)$ and $K_i(\mathcal{C}(\Omega^*, F'))$ with $K_i(\mathcal{C}_0(\Omega, F')) \times K_i(F')$ using the group isomorphism of the Alexandroff K-theorem (Theorem 2.2.1 a)) then*

$$K_i(\phi_{\Omega^*}) : K_i(\mathcal{C}(\Omega^*, F)) \longrightarrow K_i(\mathcal{C}(\Omega^*, F')), \quad (a, b) \longmapsto (K_i(\phi_{\Omega})a, K_i(\phi)b).$$

The assertion follows from Corollary 1.4.5 c). ■

COROLLARY 2.2.4 *We use the notation of the Alexandroff K-theorem (Theorem 2.2.1 a)) and denote by ω_{∞} the Alexandroff point of Ω . Let Ω' be a locally compact space,*

$$\phi' : \mathcal{C}_0(\Omega \times \Omega', F) \longrightarrow \mathcal{C}_0(\Omega^* \times \Omega', F)$$

the inclusion map, and

$$\lambda' : \mathcal{C}_0(\Omega', F) \longrightarrow \mathcal{C}_0(\Omega^* \times \Omega', F), \quad x \longmapsto \tilde{x},$$

where

$$\tilde{x} : \Omega^* \times \Omega' \longrightarrow F, \quad (\omega, \omega') \longmapsto x(\omega').$$

Then the map

$$K_i(\mathcal{C}_0(\Omega \times \Omega', F)) \times K_i(F) \longrightarrow K_i(\mathcal{C}_0(\Omega^* \times \Omega', F)),$$

$$(a, b) \longmapsto K_i(\phi')a + K_i(\lambda')b$$

is a group isomorphism.

If we put

$$\psi' : \mathcal{C}_0(\Omega^* \times \Omega', F) \longrightarrow \mathcal{C}_0(\Omega', F), \quad x \longmapsto x(\omega_\infty, \cdot)$$

then

$$0 \longrightarrow \mathcal{C}_0(\Omega \times \Omega', F) \xrightarrow{\varphi'} \mathcal{C}_0(\Omega^* \times \Omega', F) \xrightarrow[\lambda']{\psi'} \mathcal{C}_0(\Omega', F) \longrightarrow 0$$

is a split exact sequence in \mathfrak{M}_E and the assertion follows from the split exact axiom (Axiom 1.2.3). ■

2.3 Topological Sums of Locally Compact Spaces

PROPOSITION 2.3.1 (Product Theorem) *Let $(\Omega_j)_{j \in J}$ be a finite family of locally compact spaces, Ω its topological sum, and for every $j \in J$ let $\varphi_j : \mathcal{C}_0(\Omega_j, F) \longrightarrow \mathcal{C}_0(\Omega, F)$ be the inclusion map and*

$$\psi_j : \mathcal{C}_0(\Omega, F) \longrightarrow \mathcal{C}_0(\Omega_j, F), \quad x \longmapsto x|_{\Omega_j}.$$

a)

$$\Phi_i : \prod_{j \in J} K_i(\mathcal{C}_0(\Omega_j, F)) \longrightarrow K_i(\mathcal{C}_0(\Omega, F)), \quad (a_j)_{j \in J} \longmapsto \sum_{j \in J} K_i(\varphi_j) a_j$$

is a group isomorphism and

$$\Psi_i : K_i(\mathcal{C}_0(\Omega, F)) \longrightarrow \prod_{j \in J} K_i(\mathcal{C}_0(\Omega_j, F)), \quad a \longmapsto (K_i(\psi_j) a)_{j \in J}$$

is its inverse.

b) If all Ω_j , $j \in J$, belong to Υ then

$$\Omega \in \Upsilon, \quad p(\Omega) = \sum_{j \in J} p(\Omega_j), \quad q(\Omega) = \sum_{j \in J} q(\Omega_j),$$

$$\Phi_{i, \Omega, F} = \prod_{j \in J} \Phi_{i, \Omega_j, F}, \quad \bigcap_{j \in J} (\Omega_j)_\Upsilon \subset \Omega_\Upsilon.$$

c) If Ω_j is Υ -null for every $j \in J$ then Ω is also Υ -null and $\Omega^* \in \Upsilon_1$, where Ω^* denotes the Alexandroff compactification of Ω .

a) follows from Proposition 1.3.3.

b) follows from Proposition 1.5.9.

c) By b), Ω is Υ -null and by Alexandroff's K-theorem (Theorem 2.2.1 a)), $\Omega^* \in \Upsilon_1$. ■

COROLLARY 2.3.2 *Let Ω be a locally compact space, Γ a closed set of Ω , and $(\Omega_j)_{j \in J}$ a finite family of pairwise disjoint open sets of Ω such that $\bigcup_{j \in J} \Omega_j = \Omega \setminus \Gamma$. We denote for every $j \in J$ by $\varphi_j : \mathcal{C}_0(\Omega_j, F) \longrightarrow \mathcal{C}_0(\Omega, F)$ the inclusion map and assume that the maps*

$$K_i(\varphi_j) : K_i(\mathcal{C}_0(\Omega_j, F)) \longrightarrow K_i(\mathcal{C}_0(\Omega, F))$$

are group isomorphisms. If $\varphi : \mathcal{C}_0(\Omega \setminus \Gamma, F) \longrightarrow \mathcal{C}_0(\Omega, F)$ denotes the inclusion map and if we identify the above groups then $K_i(\mathcal{C}_0(\Omega \setminus \Gamma, F)) \approx K_i(\mathcal{C}_0(\Omega, F))^J$ and

$$K_i(\varphi) : K_i(\mathcal{C}_0(\Omega \setminus \Gamma, F)) \longrightarrow K_i(\mathcal{C}_0(\Omega, F)), \quad (a_j)_{j \in J} \longmapsto \sum_{j \in J} a_j. \quad \blacksquare$$

COROLLARY 2.3.3 *Let Ω be a locally compact space such that $\mathcal{C}_0(\Omega, F)$ is K-null and Γ a closed set of Ω .*

a) $K_i(\mathcal{C}_0(\Omega \setminus \Gamma, F)) \approx K_{i+1}(\mathcal{C}(\Gamma, F))$.

b) Assume Γ finite and Ω Υ -null, put

$$\psi : \mathcal{C}_0(\Omega, F) \longrightarrow \mathcal{C}(\Gamma, F), \quad x \longmapsto x|_{\Gamma},$$

and denote by $\varphi : \mathcal{C}_0(\Omega \setminus \Gamma, F) \longrightarrow \mathcal{C}_0(\Omega, F)$ the inclusion map and by δ_i the index maps associated to the exact sequence in \mathfrak{M}_E

$$0 \longrightarrow \mathcal{C}_0(\Omega \setminus \Gamma, F) \xrightarrow{\varphi} \mathcal{C}_0(\Omega, F) \xrightarrow{\psi} \mathcal{C}(\Gamma, F) \longrightarrow 0.$$

Then

$$K_i(\mathcal{C}_0(\Omega \setminus \Gamma, F)) \approx K_{i+1}(F)^{\Gamma},$$

$$\Omega \setminus \Gamma \in \Upsilon, \quad p(\Omega \setminus \Gamma) = 0, \quad q(\Omega \setminus \Gamma) = \text{Card } \Gamma, \quad \Phi_{i, (\Omega \setminus \Gamma), F} = \delta_{i+1}.$$

a) Since $\mathcal{C}_0(\Omega, F)$ is K-null, the assertion follows from the six-term axiom (Axiom 1.2.7).

b) follows from a), Lemma 2.1.4 c), and the Product Theorem (Proposition 2.3.1). ■

COROLLARY 2.3.4 *Let $(\Omega_j)_{j \in J}$ be a finite family of locally compact spaces, Ω its topological sum, and Ω^* the Alexandroff compactification of Ω .*

$$a) K_i(\mathcal{C}(\Omega, F)) \approx \prod_{j \in J} K_i(\mathcal{C}_0(\Omega_j, F)), K_i(\mathcal{C}(\Omega^*, F)) \approx K_i(F) \times \prod_{j \in J} K_i(\mathcal{C}_0(\Omega_j, F)).$$

b) *If all Ω_j , $j \in J$, belong to Υ then*

$$\Omega^* \in \Upsilon, \quad p(\Omega^*) = 1 + \sum_{j \in J} p(\Omega_j), \quad q(\Omega^*) = \sum_{j \in J} q(\Omega_j).$$

The assertion follows immediately from the Product Theorem (Proposition 2.3.1 a)) and the Alexandroff K-theorem (Theorem 2.2.1 a)). ■

COROLLARY 2.3.5 *Let $(\Omega_j)_{j \in J}$ be a finite family of locally compact spaces such that $\mathcal{C}_0(\Omega_j, F)$ is K -null for every $j \in J$ and let Γ_j be a closed set of Ω_j for every $j \in J$. We denote by Ω the Alexandroff compactification of the topological sum of the family $(\Omega_j \setminus \Gamma_j)_{j \in J}$.*

$$a) K_i(\mathcal{C}(\Omega, F)) \approx K_i(F) \times \prod_{j \in J} K_{i+1}(\mathcal{C}_0(\Gamma_j, F)).$$

b) *If for every $j \in J$, Ω_j is Υ -null and Γ_j is finite then*

$$\Omega \in \Upsilon, \quad p(\Omega) = 1, \quad q(\Omega) = \sum_{j \in J} \text{Card } \Gamma_j.$$

a) By Corollary 2.3.3 a), $K_i(\mathcal{C}_0(\Omega_j \setminus \Gamma_j, F)) \approx K_{i+1}(\mathcal{C}_0(\Gamma_j, F))$ for every $j \in J$ so by Corollary 2.3.4 a),

$$K_i(\mathcal{C}(\Omega, F)) \approx K_i(F) \times \prod_{j \in J} K_{i+1}(\mathcal{C}_0(\Gamma_j, F)).$$

b) By Corollary 2.3.3 b), for every $j \in J$,

$$\Omega_j \setminus \Gamma_j \in \Upsilon, \quad p(\Omega_j \setminus \Gamma_j) = 0, \quad q(\Omega_j \setminus \Gamma_j) = \text{Card } \Gamma_j.$$

Thus by Corollary 2.3.4 b),

$$\Omega \in \Upsilon, \quad p(\Omega) = 1, \quad q(\Omega) = \sum_{j \in J} \text{Card } \Gamma_j. \quad \blacksquare$$

PROPOSITION 2.3.6 *Let Ω be a compact space belonging to Υ_1 , Γ a closed set of Ω , $\omega_0 \in \Gamma$, and $\Gamma' := \Gamma \setminus \{\omega_0\}$. We use the notation of the Topological triple (Proposition 2.1.11) and put there*

$$\Omega_1 := \Omega, \quad \Omega_2 := \Omega \setminus \{\omega_0\}, \quad \Omega_3 := \Omega \setminus \Gamma.$$

a) $\Omega \setminus \{\omega_0\}$ is Υ -null.

b) $K_i(\mathcal{C}_0(\Omega \setminus \Gamma, F)) \approx K_{i+1}(\mathcal{C}_0(\Gamma', F))$.

c)

$$0 \longrightarrow K_i(\mathcal{C}(\Omega, F)) \xrightarrow{K_i(\psi_{1,3})} K_i(\mathcal{C}(\Gamma, F)) \xrightarrow[\leftarrow \Phi_i]{\delta_{1,3,i}} \\ \xrightarrow[\leftarrow \Phi_i]{\delta_{1,3,i}} K_{i+1}(\mathcal{C}_0(\Omega \setminus \Gamma, F)) \longrightarrow 0$$

is a split exact sequence, and the maps

$$K_i(\mathcal{C}(\Omega, F)) \times K_{i+1}(\mathcal{C}_0(\Omega \setminus \Gamma, F)) \longrightarrow K_i(\mathcal{C}(\Gamma, F)),$$

$$(a, b) \longmapsto K_i(\psi_{1,3})a + \Phi_i b,$$

$$\delta_{2,3,i} : K_i(\mathcal{C}_0(\Gamma', F)) \longrightarrow K_{i+1}(\mathcal{C}_0(\Omega \setminus \Gamma, F))$$

are group isomorphisms.

d) If $\Omega \setminus \Gamma \in \Upsilon$ or $\Gamma' \in \Upsilon$ then with the notation of Corollary 2.1.9

$$\delta_i : K_i(\mathcal{C}_0(\Gamma', F)) \longrightarrow K_{i+1}(\mathcal{C}_0(\Omega \setminus \{\omega_0\}, F))$$

is a group isomorphism and

$$\Omega \setminus \Gamma, \Gamma' \in \Upsilon, \quad p(\Omega \setminus \Gamma) = q(\Gamma'), \quad q(\Omega \setminus \Gamma) = p(\Gamma'),$$

$$\Phi_{i,(\Omega \setminus \Gamma),F} = \delta_{i+1} \circ \Phi_{(i+1),\Gamma',F}.$$

e) Assume Γ finite.

$e_1) (\delta_{2,3,i})^{-1} : K_{i+1}(\mathcal{C}_0(\Omega \setminus \Gamma, F)) \longrightarrow K_i(F)^{\Gamma'}$ is a group isomorphism.

$e_2) \Omega \setminus \Gamma \in \Upsilon, \quad p(\Omega \setminus \Gamma) = 0, \quad q(\Omega \setminus \Gamma) = \text{Card } \Gamma'.$

a) follows from Alexandroff's K-theorem (Theorem 2.2.1 c)).

b) follows from Corollary 2.3.3 a).

c) By a), $\Omega \setminus \{\omega\}$ is K-null and the assertion follows from the Topological triple (Proposition 2.1.11 a)).

d) follows from Corollary 2.1.9.

e_1) follows from c) and the Product Theorem (Proposition 2.3.1 a_4)).

e_2) follows from a) and Corollary 2.1.9 c). ■

PROPOSITION 2.3.7 *Let Ω be a locally compact space, Γ a closed set of Ω , $\varphi : \mathcal{C}_0(\Omega \setminus \Gamma, F) \longrightarrow \mathcal{C}_0(\Omega, F)$ the inclusion map,*

$$\psi : \mathcal{C}_0(\Omega, F) \longrightarrow \mathcal{C}_0(\Gamma, F), \quad x \longmapsto x|_{\Gamma},$$

and δ_i the index maps associated to the exact sequence in \mathfrak{M}_E

$$0 \longrightarrow \mathcal{C}_0(\Omega \setminus \Gamma, F) \xrightarrow{\varphi} \mathcal{C}_0(\Omega, F) \xrightarrow{\psi} \mathcal{C}_0(\Gamma, F) \longrightarrow 0.$$

Let $(\Omega_j)_{j \in J}$ be a finite family of pairwise disjoint open sets of Ω the union of which is $\Omega \setminus \Gamma$ and for every $j \in J$ put

$$\psi_j : \mathcal{C}_0(\bar{\Omega}_j, F) \longrightarrow \mathcal{C}_0(\bar{\Omega}_j \setminus \Omega_j, F), \quad x \longmapsto x|_{(\bar{\Omega}_j \setminus \Omega_j)},$$

$$\psi'_j : \mathcal{C}_0(\Omega \setminus \Gamma, F) \longrightarrow \mathcal{C}_0(\Omega_j, F), \quad x \longmapsto x|_{\Omega_j},$$

$$\psi''_j : \mathcal{C}_0(\Gamma, F) \longrightarrow \mathcal{C}_0(\bar{\Omega}_j \setminus \Omega_j, F), \quad x \longmapsto x|_{(\bar{\Omega}_j \setminus \Omega_j)}$$

and denote by

$$\varphi_j : \mathcal{C}_0(\Omega_j, F) \longrightarrow \mathcal{C}_0(\bar{\Omega}_j, F),$$

$$\varphi'_j : \mathcal{C}_0(\Omega_j, F) \longrightarrow \mathcal{C}_0(\Omega \setminus \Gamma, F),$$

$$\varphi''_j : \mathcal{C}_0(\Omega_j, F) \longrightarrow \mathcal{C}_0(\Omega, F)$$

the inclusion maps and by $\delta_{j,i}$ the index maps associated to the exact sequence in \mathfrak{M}_E

$$0 \longrightarrow \mathcal{C}_0(\Omega_j, F) \xrightarrow{\varphi_j} \mathcal{C}_0(\bar{\Omega}_j, F) \xrightarrow{\psi_j} \mathcal{C}_0(\bar{\Omega}_j \setminus \Omega_j, F) \longrightarrow 0.$$

a) For every $j \in J$,

$$\delta_{j,i} \circ K_i(\psi_j'') = K_{i+1}(\psi_j') \circ \delta_i$$

and

$$\delta_i = \sum_{j \in J} K_{i+1}(\varphi_j') \circ \delta_{j,i} \circ K_i(\psi_j'') .$$

b) $K_i(\varphi) = \sum_{j \in J} K_i(\varphi_j'') \circ K_i(\psi_j')$.

c) Let $j_0 \in J$ such that $\mathcal{C}_0(\Omega \setminus \Omega_{j_0}, F)$ is K -null.

$c_1)$ $K_i(\varphi_{j_0}'')$ is a group isomorphism.

$c_2)$ Assume ψ K -null. If we put

$$\Phi_i := K_i(\varphi_{j_0}') \circ K_i(\varphi_{j_0}'')^{-1} : K_i(\mathcal{C}_0(\Omega, F)) \longrightarrow K_i(\mathcal{C}_0(\Omega \setminus \Gamma, F))$$

then

$$\begin{array}{c} 0 \longrightarrow K_{i+1}(\mathcal{C}_0(\Gamma, F)) \xrightarrow{\delta_{i+1}} K_i(\mathcal{C}_0(\Omega \setminus \Gamma, F)) \xrightarrow[\Phi_i]{K_i(\varphi)} \\ \xleftarrow[\Phi_i]{K_i(\varphi)} K_i(\mathcal{C}_0(\Omega, F)) \longrightarrow 0 \end{array}$$

is a split exact sequence and the map

$$K_{i+1}(\mathcal{C}_0(\Gamma, F)) \times K_i(\mathcal{C}_0(\Omega, F)) \longrightarrow K_i(\mathcal{C}_0(\Omega \setminus \Gamma, F)),$$

$$(a, b) \longmapsto \delta_{i+1}a + \Phi_i b$$

is a group isomorphism.

a) By the commutativity of the index maps (Axiom 1.2.8),

$$\delta_{j,i} \circ K_i(\psi_j'') = K_{i+1}(\psi_j') \circ \delta_i .$$

Since $\sum_{j \in J} \varphi_j' \circ \psi_j'$ is the identity map of $\mathcal{C}_0(\Omega \setminus \Gamma, F)$,

$$\begin{aligned} \sum_{j \in J} K_{i+1}(\varphi_j') \circ \delta_{j,i} \circ K_i(\psi_j'') &= \sum_{j \in J} K_{i+1}(\varphi_j') \circ K_{i+1}(\psi_j') \circ \delta_i = \\ &= K_{i+1} \left(\sum_{j \in J} \varphi_j' \circ \psi_j' \right) \circ \delta_i = \delta_i . \end{aligned}$$

b) We have $\varphi_j'' = \varphi \circ \varphi_j'$ for every $j \in J$. Since $\sum_{j \in J} \varphi_j' \circ \psi_j'$ is the identity map of $\mathcal{C}_0(\Omega \setminus \Gamma, F)$,

$$\begin{aligned} K_i(\varphi) &= K_i(\varphi) \circ K_i\left(\sum_{j \in J} \varphi_j' \circ \psi_j'\right) = \\ &= \sum_{j \in J} K_i(\varphi) \circ K_i(\varphi_j') \circ K_i(\psi_j') = \sum_{j \in J} K_i(\varphi_j'') \circ K_i(\psi_j'). \end{aligned}$$

c_1) If we put

$$\bar{\psi} : \mathcal{C}_0(\Omega, F) \longrightarrow \mathcal{C}_0(\Omega \setminus \Omega_{j_0}, F), \quad x \longmapsto x|(\Omega \setminus \Omega_{j_0})$$

then

$$0 \longrightarrow \mathcal{C}_0(\Omega_{j_0}, F) \xrightarrow{\varphi_{j_0}''} \mathcal{C}_0(\Omega, F) \xrightarrow{\bar{\psi}} \mathcal{C}_0(\Omega \setminus \Omega_{j_0}, F) \longrightarrow 0$$

is an exact sequence in \mathfrak{M}_E . Since $\mathcal{C}_0(\Omega \setminus \Omega_{j_0}, F)$ is K-null, it follows that $K_i(\varphi_{j_0}'')$ is a group isomorphism by the Topological six-term sequence (Proposition 2.1.8 c_1)).

c_2) Since $\varphi \circ \varphi_{j_0}' = \varphi_{j_0}''$,

$$K_i(\varphi) \circ \Phi_i = K_i(\varphi) \circ K_i(\varphi_{j_0}') \circ K_i(\varphi_{j_0}'')^{-1} = K_i(\varphi_{j_0}'') \circ K_i(\varphi_{j_0}'')^{-1} = id_{K_i(\mathcal{C}_0(\Omega, F))}.$$

Since ψ is K-null,

$$0 \longrightarrow K_{i+1}(\mathcal{C}_0(\Gamma, F)) \xrightarrow{\delta_{i+1}} K_i(\mathcal{C}_0(\Omega \setminus \Gamma, F)) \xleftarrow[\Phi_i]{K_i(\varphi)} K_i(\mathcal{C}_0(\Omega, F)) \longrightarrow 0$$

is a split exact sequence and this implies the last assertion. ■

PROPOSITION 2.3.8 *If $(\Omega_j)_{j \in J}$, $J \neq \emptyset$, is a finite family of compact spaces belonging to Υ_1 then $\prod_{j \in J} \Omega_j \in \Upsilon_1$.*

The assertion follows immediately from Proposition 1.6.5. ■

2.4 Homotopy

PROPOSITION 2.4.1 *Let Ω be a locally compact space, Ω^* its Alexandroff compactification, $(\vartheta_s)_{s \in [0,1]}$ a family of proper continuous maps $\Omega \rightarrow \Omega$, and for every $s \in [0,1]$ let $\vartheta_s^* : \Omega^* \rightarrow \Omega^*$ be the continuous extension of ϑ_s . We assume:*

- 1) $\Omega^* \times]0, 1] \longrightarrow \Omega^*$, $(\omega, s) \longmapsto \vartheta_s^*(\omega)$ is continuous,
- 2) $\vartheta_1(\omega) = \omega$ for every $\omega \in \Omega$,
- 3) for every compact set Γ of Ω there is an $\varepsilon \in]0, 1]$ with $\Gamma \cap \vartheta_s(\Omega) = \emptyset$ for all $s \in]0, \varepsilon[$.

Then Ω is null-homotopic and $\Omega^* \in \Upsilon_1$.

We put for every $s \in [0, 1]$,

$$\phi_s : \mathcal{C}_0(\Omega, \mathbb{C}) \longrightarrow \mathcal{C}_0(\Omega, \mathbb{C}), \quad x \longmapsto \begin{cases} x \circ \vartheta_s & \text{if } s \in]0, 1] \\ 0 & \text{if } s = 0 \end{cases}.$$

Then $(\phi_s)_{s \in [0, 1]}$ is a pointwise continuous path in $\mathcal{C}_0(\Omega, \mathbb{C})$ with $\phi_0 = 0$ and ϕ_1 the identity map of $\mathcal{C}_0(\Omega, F)$. Thus Ω is null-homotopic. By Proposition 1.5.4 d), Ω is Υ -null and by Alexandroff's K-theorem, (Theorem 2.2.1 c)), $\Omega^* \in \Upsilon_1$. ■

COROLLARY 2.4.2 *Let J be a set and $\Omega := [0, 1]^J$. Then $\Omega \setminus \{0\}$ is null-homotopic and $\Omega \in \Upsilon_1$.*

The assertion follows from Proposition 2.4.1 by using the map

$$\vartheta : \Omega \times [0, 1] \longrightarrow \Omega, \quad (\omega, s) \longmapsto s\omega. \quad \blacksquare$$

PROPOSITION 2.4.3 *Let Ω be a locally compact space, Γ_0, Γ_1 compact subspaces of Ω , $\vartheta_0 : \Gamma_0 \longrightarrow \Gamma_1$ a homeomorphism, and $\vartheta : \Gamma_0 \times [0, 1] \longrightarrow \Omega$ a continuous map such that $\vartheta(\omega, 0) = \omega$ and $\vartheta(\omega, 1) = \vartheta_0(\omega)$ for every $\omega \in \Gamma_0$. We put*

$$\psi_j : \mathcal{C}_0(\Omega, F) \longrightarrow \mathcal{C}(\Gamma_j, F), \quad x \longmapsto x|_{\Gamma_j}$$

for every $j \in \{0, 1\}$ and

$$\varphi : \mathcal{C}(\Gamma_1, F) \longrightarrow \mathcal{C}(\Gamma_0, F), \quad x \longmapsto x \circ \vartheta_0.$$

- a) $K_i(\varphi)$ is a group isomorphism and $K_i(\psi_0) = K_i(\varphi) \circ K_i(\psi_1)$.

b) For every $j \in \{0, 1\}$ let $\varphi_j : \mathcal{C}_0(\Omega \setminus \Gamma_j, F) \longrightarrow \mathcal{C}_0(\Omega, F)$ be the inclusion map and $\mathcal{C}(\Gamma_j, F) \xrightarrow{\lambda_j} \mathcal{C}_0(\Omega, F)$ be a morphism in \mathfrak{M}_E such that $\psi_j \circ \lambda_j = \text{id}_{\mathcal{C}(\Gamma_j, F)}$ and $\lambda_1 = \lambda_0 \circ \varphi$.

b₁) For every $j \in \{0, 1\}$,

$$\begin{array}{c} 0 \longrightarrow K_i(\mathcal{C}_0(\Omega \setminus \Gamma_j, F)) \xrightarrow{K_i(\varphi_j)} K_i(\mathcal{C}_0(\Omega, F)) \xleftarrow{\begin{array}{c} K_i(\psi_j) \\ K_i(\lambda_j) \end{array}} \\ \xleftarrow{\begin{array}{c} K_i(\psi_j) \\ K_i(\lambda_j) \end{array}} K_i(\mathcal{C}_0(\Gamma_j, F)) \longrightarrow 0 \end{array}$$

is a split exact sequence.

b₂) $\text{Im } K_i(\varphi_0) = \text{Im } K_i(\varphi_1)$.

b₃) If we put for every $j \in \{0, 1\}$

$$\Psi_{j,i} : K_i(\mathcal{C}_0(\Omega \setminus \Gamma_j, F)) \longrightarrow \text{Im } K_i(\varphi_j), \quad a \longmapsto K_i(\varphi_j)a$$

then $\Psi_{j,i}$ and

$$(\Psi_{1,i})^{-1} \circ \Psi_{0,i} : K_i(\mathcal{C}_0(\Omega \setminus \Gamma_0, F)) \longrightarrow K_i(\mathcal{C}_0(\Omega \setminus \Gamma_1, F))$$

are well-defined group isomorphisms.

b₄) If $\Omega \setminus \Gamma_0 \in \Upsilon$ or $\Omega \setminus \Gamma_1 \in \Upsilon$ then

$$\Omega \setminus \Gamma_0, \Omega \setminus \Gamma_1 \in \Upsilon, \quad p(\Omega \setminus \Gamma_0) = p(\Omega \setminus \Gamma_1), \quad q(\Omega \setminus \Gamma_0) = q(\Omega \setminus \Gamma_1),$$

$$\Phi_{i,(\Omega \setminus \Gamma_1),F} = (\Psi_{1,i}^F)^{-1} \circ \Psi_{0,i}^F \circ \Phi_{i,(\Omega \setminus \Gamma),F}.$$

c) If Ω is compact and if for every $j \in \{0, 1\}$ there is a continuous map $\vartheta'_j : \Omega \longrightarrow \Gamma_j$ such that $\vartheta'_j(\omega) = \omega$ for every $\omega \in \Gamma_j$ and $\vartheta_0 \circ \vartheta'_0 = \vartheta'_1$ then the hypotheses of b) are fulfilled.

a) For every $s \in [0, 1]$ put

$$v_s : \mathcal{C}_0(\Omega, F) \longrightarrow \mathcal{C}(\Gamma_0, F), \quad x \longmapsto x(\vartheta(\cdot, s)).$$

Then $K_i(v_0) = K_i(v_1)$ by the homotopy axiom (Axiom 1.2.5). $K_i(\varphi)$ is obviously a group isomorphism. For every $x \in \mathcal{C}_0(\Omega, F)$ and $\omega \in \Gamma_0$,

$$(v_0 x)(\omega) = x(\vartheta(\omega, 0)) = x(\omega) = (\psi_0 x)(\omega),$$

$$(\nu_1 x)(\omega) = x(\vartheta(\omega, 1)) = x(\vartheta_0(\omega)) = (\psi_1 x)(\vartheta_0(\omega)) = (\varphi \psi_1 x)(\omega),$$

so $\nu_0 = \psi_0$, $\nu_1 = \varphi \circ \psi_1$,

$$K_i(\psi_0) = K_i(\nu_0) = K_i(\nu_1) = K_i(\varphi) \circ K_i(\psi_1).$$

b_1 follows from the split exact axiom (Axiom 1.2.3).

b_2) Let $j \in \{0, 1\}$. We want to prove

$$\text{Im } K_i(\varphi_j) = \{ c - K_i(\lambda_j) K_i(\psi_j) c \mid c \in K_i(\mathcal{C}(\Omega, F)) \}.$$

Let $a \in K_i(\mathcal{C}_0(\Omega \setminus \Gamma_j, F))$ and put $c := K_i(\varphi_j) a$. Then

$$c - K_i(\lambda_j) K_i(\psi_j) c = K_i(\varphi_j) a - K_i(\lambda_j) K_i(\psi_j) K_i(\varphi_j) a = K_i(\varphi_j) a,$$

which proves the " \subset "-inclusion. Let $c \in K_i(\mathcal{C}(\Omega, F))$. Then

$$\begin{aligned} & K_i(\psi_j)(c - K_i(\lambda_j) K_i(\psi_j) c) = \\ &= K_i(\psi_j) c - K_i(\psi_j) K_i(\lambda_j) K_i(\psi_j) c = K_i(\psi_j) c - K_i(\psi_j) c = 0, \\ & c - K_i(\lambda_j) K_i(\psi_j) c \in \text{Ker } K_i(\psi_j) = \text{Im } K_i(\varphi_j), \end{aligned}$$

which proves the " \supset "-inclusion (by b_1)).

Since $\lambda_1 \circ \psi_1 = \lambda_0 \circ \varphi \circ \psi_1$, we get by a),

$$K_i(\lambda_1) \circ K_i(\psi_1) = K_i(\lambda_0) \circ K_i(\varphi) \circ K_i(\psi_1) = K_i(\lambda_0) \circ K_i(\psi_0).$$

Thus, by the above, $\text{Im } K_i(\varphi_0) = \text{Im } K_i(\varphi_1)$.

b_3) By b_1), $K_i(\varphi_0)$ and $K_i(\varphi_1)$ are injective, the assertion follows from b_2).

b_4) Let $F \xrightarrow{\phi} F'$ be a morphism in \mathfrak{M}_E and for every $j \in \{0, 1\}$ put

$$\mu_j : \mathcal{C}_0(\Omega \setminus \Gamma_j, F) \longrightarrow \mathcal{C}_0(\Omega \setminus \Gamma_j, F'), \quad x \longmapsto \phi \circ x,$$

$$\mu : \mathcal{C}_0(\Omega, F) \longrightarrow \mathcal{C}_0(\Omega, F'), \quad x \longmapsto \phi \circ x.$$

We mark by a prime the notation associated to F when applied to F' . For every $j \in \{0, 1\}$ the diagram

$$\begin{array}{ccc} \mathcal{C}_0(\Omega \setminus \Gamma_j, F) & \xrightarrow{\mu_j} & \mathcal{C}_0(\Omega \setminus \Gamma_j, F') \\ \varphi_j \downarrow & & \downarrow \varphi'_j \\ \mathcal{C}_0(\Omega, F) & \xrightarrow{\mu} & \mathcal{C}_0(\Omega, F') \end{array}$$

is commutative. Thus the diagrams

$$\begin{array}{ccc} K_i(\mathcal{C}_0(\Omega \setminus \Gamma_j, F)) & \xrightarrow{K_i(\mu_j)} & K_i(\mathcal{C}_0(\Omega \setminus \Gamma_j, F')) \\ K_i(\varphi_j) \downarrow & & \downarrow K_i(\varphi'_j) \\ K_i(\mathcal{C}_0(\Omega, F)) & \xrightarrow{K_i(\mu)} & K_i(\mathcal{C}_0(\Omega, F')) \end{array}$$

$$\begin{array}{ccc} K_i(\mathcal{C}_0(\Omega \setminus \Gamma_j, F)) & \xrightarrow{K_i(\mu_j)} & K_i(\mathcal{C}_0(\Omega \setminus \Gamma_j, F')) \\ \Psi_{j,i} \downarrow & & \downarrow \Psi'_{j,i} \\ \text{Im } K_i(\varphi_j) & \xrightarrow{\Lambda_i} & \text{Im } K_i(\varphi'_j) \end{array}$$

are also commutative, where Λ_i is the map defined by $K_i(\mu)$.

Assume $\Omega \setminus \Gamma_0 \in \Upsilon$ and consider the diagram (by b_2))

$$\begin{array}{ccc} K_i(F)^{p(\Omega \setminus \Gamma_0)} \times K_{i+1}(F)^{q(\Omega \setminus \Gamma_0)} & \xrightarrow{\Delta} & A \\ \Phi_{i,(\Omega \setminus \Gamma_0),F} \downarrow & & \downarrow \Phi_{i,(\Omega \setminus \Gamma_0),F'} \\ K_i(\mathcal{C}_0(\Omega \setminus \Gamma_0, F)) & \xrightarrow{K_i(\mu_0)} & K_i(\mathcal{C}_0(\Omega \setminus \Gamma_0, F')) \\ \Psi_{0,i} \downarrow & & \downarrow \Psi'_{0,i} \\ \text{Im } K_i(\varphi_0) & \xrightarrow{\Lambda_i} & \text{Im } K_i(\varphi'_0) \\ \Psi_{1,i} \uparrow & & \uparrow \Psi'_{1,i} \\ K_i(\mathcal{C}_0(\Omega \setminus \Gamma_1, F)) & \xrightarrow{K_i(\mu_1)} & K_i(\mathcal{C}_0(\Omega \setminus \Gamma_1, F')) \end{array}$$

where

$$\begin{aligned} \Delta &:= K_i(\phi)^{p(\Omega \setminus \Gamma_0)} \times K_{i+1}(\phi)^{q(\Omega \setminus \Gamma_0)}, \\ A &:= K_i(F')^{p(\Omega \setminus \Gamma_0)} \times K_{i+1}(F')^{q(\Omega \setminus \Gamma_0)}. \end{aligned}$$

By the above, this diagram is commutative and the assertion follows from b_3).

c) For every $j \in \{0, 1\}$ put

$$\lambda_j : \mathcal{C}(\Gamma_j, F) \longrightarrow \mathcal{C}(\Omega, F), \quad x \longmapsto x \circ \vartheta'_j.$$

Then $\psi_j \circ \lambda_j = id_{\mathcal{C}(\Gamma_j, F)}$ and for every $x \in \mathcal{C}(\Gamma_1, F)$,

$$\lambda_1 x = x \circ \vartheta'_1 = x \circ \vartheta_0 \circ \vartheta'_0 = (\varphi x) \circ \vartheta'_0 = \lambda_0(\varphi x), \quad \lambda_1 = \lambda_0 \circ \varphi. \quad \blacksquare$$

COROLLARY 2.4.4 *Let Ω be a compact space and $\omega, \omega' \in \Omega$ such that there is a continuous path in Ω from ω to ω' .*

a) $K_i(\mathcal{C}_0(\Omega \setminus \{\omega\}, F)) \approx K_i(\mathcal{C}_0(\Omega \setminus \{\omega'\}, F))$.

b) If $\Omega \setminus \{\omega\} \in \Upsilon$ then

$$\Omega \setminus \{\omega'\} \in \Upsilon, \quad p(\Omega \setminus \{\omega'\}) = p(\Omega \setminus \{\omega\}), \quad q(\Omega \setminus \{\omega'\}) = q(\Omega \setminus \{\omega\}).$$

a) follows from Proposition 2.4.3 b_3) and c).

b) follows from Proposition 2.4.3 b_4) and c). \blacksquare

COROLLARY 2.4.5 *Let Ω, Ω' be compact spaces such that $\Omega' \setminus \{\omega'\}$ is null-homotopic for all $\omega' \in \Omega'$, $\omega \in \Omega$, and $\omega'' \in \Omega \times \Omega'$. Then*

$$\begin{aligned} K_i(\mathcal{C}_0(\Omega \setminus \{\omega\}, F)) &\approx K_i(\mathcal{C}_0((\Omega \setminus \{\omega\}) \times \Omega', F)) \approx \\ &\approx K_i(\mathcal{C}_0(\Omega \times \Omega' \setminus \{\omega''\}, F)). \end{aligned}$$

Let $\omega'' = (\omega_0, \omega'_0) \in \Omega \times \Omega'$. By Corollary 2.1.10 a),

$$K_i(\mathcal{C}_0((\Omega \setminus \{\omega_0\}) \times \Omega', F)) \approx K_i(\mathcal{C}_0(\Omega \times \Omega' \setminus \{\omega''\}, F))$$

and by Proposition 2.4.3 c),

$$K_i(\mathcal{C}_0((\Omega \setminus \{\omega\}) \times \Omega', F)) \approx K_i(\mathcal{C}_0((\Omega \setminus \{\omega_0\}) \times \Omega', F)).$$

By Proposition 1.4.2 b_3), c),

$$\mathcal{C}_0((\Omega \setminus \{\omega\}) \times (\Omega' \setminus \{\omega'_0\}), F) \approx \mathcal{C}_0(\Omega' \setminus \{\omega'_0\}, \mathbf{C}) \otimes \mathcal{C}_0(\Omega \setminus \{\omega\}, F)$$

is null-homotopic. Since the sequence in \mathfrak{M}_E

$$\begin{aligned} 0 \longrightarrow \mathcal{C}_0((\Omega \setminus \{\omega\}) \times (\Omega' \setminus \{\omega'_0\}), F) &\longrightarrow \mathcal{C}_0((\Omega \setminus \{\omega\}) \times \Omega', F) \\ \mathcal{C}_0((\Omega \setminus \{\omega\}) \times \Omega', F) &\longrightarrow \mathcal{C}_0((\Omega \setminus \{\omega\}) \times \{\omega'_0\}, F) \longrightarrow 0 \end{aligned}$$

is exact it follows from the topological six-term sequence (Proposition 2.1.8 a_1)),

$$\begin{aligned} K_i(\mathcal{C}_0((\Omega \setminus \{\omega\}) \times \Omega', F)) &\approx \\ \approx K_i(\mathcal{C}_0((\Omega \setminus \{\omega\}) \times \{\omega'_0\}, F)) &\approx K_i(\mathcal{C}_0(\Omega \setminus \{\omega\}, F)) . \end{aligned}$$

■

COROLLARY 2.4.6 *Let Ω be a locally compact space and $\omega_1, \omega_2 \in \Omega$ and for every $j \in \{1, 2\}$ put*

$$\psi_j : \mathcal{C}_0(\Omega, F) \longrightarrow F, \quad x \longmapsto x(\omega_j) .$$

If there is a continuous path in Ω from ω_1 to ω_2 then $K_i(\psi_1) = K_i(\psi_2)$.

The assertion follows from Proposition 2.4.3 a). ■

COROLLARY 2.4.7 *Let Ω be a locally compact space, Γ a finite subset of Ω , $\omega_0 \in \Omega$, and*

$$\begin{aligned} \psi : \mathcal{C}_0(\Omega, F) &\longrightarrow \mathcal{C}(\Gamma, F), \quad x \longmapsto x|_{\Gamma}, \\ \psi_{\omega_0} : \mathcal{C}_0(\Omega, F) &\longrightarrow F, \quad x \longmapsto x(\omega_0) . \end{aligned}$$

If for every $\omega \in \Gamma$ there is a continuous path in Ω connecting ω_0 with ω then

$$\begin{aligned} K_i(\psi) : K_i(\mathcal{C}_0(\Omega, F)) &\longrightarrow K_i(\mathcal{C}(\Gamma, F)) \approx K_i(F)^{Card \Gamma}, \\ a &\longmapsto (K_i(\psi_{\omega_0})a)_{\omega \in \Gamma} . \end{aligned}$$

We put

$$\psi_{\omega} : \mathcal{C}_0(\Omega, F) \longrightarrow F, \quad x \longmapsto x(\omega)$$

for every $\omega \in \Gamma$. By Corollary 2.4.6, $K_i(\psi_{\omega}) = K_i(\psi_{\omega_0})$ for every $\omega \in \Gamma$ and the assertion follows from the Product Theorem (Proposition 2.3.1 a). ■

PROPOSITION 2.4.8 *Let Ω be a path connected compact space, Γ a finite subset of Ω , $\omega_0 \in \Gamma$, $\Gamma' := \Gamma \setminus \{\omega_0\}$,*

$$\begin{aligned}\varphi : \mathcal{C}_0(\Omega \setminus \Gamma, F) &\longrightarrow \mathcal{C}(\Omega, F), \\ \varphi' : \mathcal{C}_0(\Omega \setminus \Gamma, F) &\longrightarrow \mathcal{C}_0(\Omega \setminus \{\omega_0\}, F), \\ \varphi'' : \mathcal{C}(\Gamma', F) &\longrightarrow \mathcal{C}(\Gamma, F)\end{aligned}$$

the inclusion maps,

$$\begin{aligned}\psi : \mathcal{C}(\Omega, F) &\longrightarrow \mathcal{C}(\Gamma, F), \quad x \longmapsto x|_{\Gamma}, \\ \psi' : \mathcal{C}_0(\Omega \setminus \{\omega_0\}, F) &\longrightarrow \mathcal{C}(\Gamma', F), \quad x \longmapsto x|_{\Gamma'}, \\ \psi_{\omega} : \mathcal{C}(\Omega, F) &\longrightarrow F, \quad x \longmapsto x(\omega)\end{aligned}$$

for every $\omega \in \Gamma$, and δ_i, δ'_i the index maps associated to the exact sequences in \mathfrak{M}_E

$$\begin{aligned}0 \longrightarrow \mathcal{C}_0(\Omega \setminus \Gamma, F) &\xrightarrow{\varphi} \mathcal{C}(\Omega, F) \xrightarrow{\psi} \mathcal{C}(\Gamma, F) \longrightarrow 0, \\ 0 \longrightarrow \mathcal{C}_0(\Omega \setminus \Gamma, F) &\xrightarrow{\varphi'} \mathcal{C}_0(\Omega \setminus \{\omega_0\}, F) \xrightarrow{\psi'} \mathcal{C}(\Gamma', F) \longrightarrow 0.\end{aligned}$$

a) $K_i(\mathcal{C}(\Omega, F)) \approx K_i(F) \times K_i(\mathcal{C}_0(\Omega \setminus \{\omega_0\}, F))$.

b) ψ' is K -null.

c) *If we use the group isomorphism of a) then*

$$K_i(\psi) : K_i(\mathcal{C}(\Omega, F)) \longrightarrow K_i(\mathcal{C}(\Gamma, F)) \approx K_i(F)^{\Gamma}, \quad (a, b) \longmapsto (a)_{\omega \in \Gamma}.$$

d) *If we identify $K_i(\mathcal{C}(\Gamma, F))$ with $K_i(F)^{\Gamma}$ and $K_i(\mathcal{C}(\Gamma', F))$ with $K_i(F)^{\Gamma'}$ then*

$$\delta_i : K_i(\mathcal{C}(\Gamma, \cdot)) \longrightarrow K_{i+1}(\mathcal{C}_0(\Omega \setminus \Gamma, \cdot)), \quad (a_{\omega})_{\omega \in \Gamma} \longmapsto (\delta'_i(a_{\omega} - a_{\omega_0}))_{\omega \in \Gamma'}.$$

e) *Assume $\mathcal{C}_0(\Omega \setminus \{\omega_0\}, F)$ K -null.*

e₁) $K_i(\psi_{\omega_0}) : K_i(\mathcal{C}_0(\Omega, F)) \longrightarrow K_i(F)$ *is a group isomorphism.*

e₂) $\delta'_i : K_i(\mathcal{C}(\Gamma', F)) \longrightarrow K_{i+1}(\mathcal{C}_0(\Omega \setminus \Gamma, F))$ *is a group isomorphism.*

e₃) *If we identify $K_i(\mathcal{C}(\Gamma', F))$ with $K_i(F)^{\Gamma'}$ and $K_i(\mathcal{C}(\Gamma, F))$ with $K_i(F)^{\Gamma}$ then for all $(a_{\omega})_{\omega \in \Gamma'}$*

$$K_i(\varphi'')(a_{\omega})_{\omega \in \Gamma'} = (a_{\omega})_{\omega \in \Gamma},$$

where $a_{\omega_0} = 0$.

e_4) If we identify $K_{i+1}(\mathcal{C}_0(\Omega \setminus \Gamma, F))$ with $K_i(\mathcal{C}(\Gamma', F))$ using $(\delta'_i)^{-1}$ of e_2) then for all $(a_\omega)_{\omega \in \Gamma} \in K_i(\mathcal{C}(\Gamma, F))$,

$$\delta_i(a_\omega)_{\omega \in \Gamma} = (a_\omega - a_{\omega_0})_{\omega \in \Gamma'}.$$

a) follows from the Alexandroff K-theorem (Theorem 2.2.1 a)).

b) Let $\omega \in \Gamma'$ and let $\vartheta : [0, 1] \longrightarrow \Omega$ be a continuous path in Ω connecting ω with ω_0 . Then for every $x \in \mathcal{C}_0(\Omega \setminus \{\omega_0\}, F)$ the map

$$[0, 1] \longrightarrow \mathcal{C}_0(\Omega \setminus \{\omega_0\}, F), \quad x \longmapsto x(\vartheta_s(\omega))$$

is continuous. By the homotopy axiom (Axiom 1.2.5), $K_i(\psi_\omega) = 0$ so by the Product Theorem (Proposition 2.3.1 a)), $K_i(\psi') = 0$.

c) follows from a), b), and Corollary 2.4.7.

d) By the commutativity of the index maps (Axiom 1.2.8), $\delta'_i = \delta_i \circ K_i(\varphi'')$ so by the Product Theorem (Proposition 2.3.1 a)),

$$\delta_i(0, (a_\omega)_{\omega \in \Gamma'}) = \delta'_i(a_\omega)_{\omega \in \Gamma'}$$

for all $(a_\omega)_{\omega \in \Gamma'} \in K_i(F)^{\Gamma'}$. For $a \in K_i(F)$, by c) and by the above,

$$\begin{aligned} 0 &= \delta_i K_i(\psi) a = \delta_i(a)_{\omega \in \Gamma} = \delta_i(a, (a)_{\omega \in \Gamma'}) = \\ &= \delta_i(a, 0) + \delta_i(0, (a)_{\omega \in \Gamma'}) = \delta_i(a, 0) + \delta'_i(a)_{\omega \in \Gamma'}, \end{aligned}$$

$\delta_i(a, 0) = -\delta'_i(a)_{\omega \in \Gamma'}$. It follows for all $(a_\omega)_{\omega \in \Gamma}$,

$$\begin{aligned} \delta_i(a_\omega)_{\omega \in \Gamma} &= \delta_i(a_{\omega_0}, 0) + \delta_i(0, (a_\omega)_{\omega \in \Gamma'}) = \\ &= -\delta'_i(a_{\omega_0})_{\omega \in \Gamma'} + \delta'_i(a_\omega)_{\omega \in \Gamma'} = \delta'_i(a_\omega - a_{\omega_0})_{\omega \in \Gamma}. \end{aligned}$$

e_1) and e_2) follow from the Topological six-term sequence (Proposition 2.1.8) a_1) and b_1), respectively.

e_3) follows from the Product Theorem (Proposition 2.3.1 a)).

e_4) follows from d). ■

EXAMPLE 2.4.9 Let $n \in \mathbb{N}$. We use the notation of Proposition 2.4.8 and put

$$\Omega := \left\{ re^{\frac{2\pi i j}{n}} \mid r \in [0, 1], j \in \mathbb{N}_n \right\}, \quad \Gamma := \left\{ e^{\frac{2\pi i j}{n}} \mid j \in \mathbb{N}_n \right\}, \quad \omega_0 := 1.$$

- a) $\Omega \setminus \{\omega_0\}$ is null-homotopic and so K -null.
- b) $K_i(\psi_{\omega_0}) : K_i(\mathcal{C}_0(\Omega, F)) \longrightarrow K_i(F)$ is a group isomorphism.
- c) $\delta'_i : K_i(\mathcal{C}(\Gamma', F)) \approx K_i(F)^{\Gamma'} \longrightarrow K_{i+1}(\mathcal{C}_0(\Omega \setminus \Gamma, F))$ is a group isomorphism.
- d) If we identify $K_i(\mathcal{C}(\Gamma', F))$ with $K_i(F)^{\Gamma'}$ and $K_i(\mathcal{C}(\Gamma, F))$ with $K_i(F)^\Gamma$ (using e.g. Lemma 2.1.4 c)) then for all $(a_\omega)_{\omega \in \Gamma'}$

$$K_i(\phi'')(a_\omega)_{\omega \in \Gamma'} = (a_\omega)_{\omega \in \Gamma},$$

where $a_{\omega_0} = 0$.

- e) If we identify $K_{i+1}(\mathcal{C}_0(\Omega \setminus \Gamma, F))$ with $K_i(F)^{\Gamma'}$ using $(\delta'_i)^{-1}$ of c) then for all $(a_\omega)_{\omega \in \Gamma}$,

$$\delta_i(a_\omega)_{\omega \in \Gamma} = (a_\omega - a_{\omega_0})_{\omega \in \Gamma'}.$$

- f) $\Omega \in \Upsilon$, $p(\Omega) = 1$, $q(\Omega) = 0$, $\Phi_{i, \Omega, F} = K_i(\psi_{\omega_0})$, $\Omega_\Upsilon = \mathbb{C}_\Upsilon$.

- a) By Proposition 2.4.1, $\Omega \setminus \{\omega_0\}$ is null-homotopic.
- b) follows from a) and the Topological six-term sequence (Proposition 2.1.8 a)).
- c), d), and e) follow from Proposition 2.4.8 b), c), and d), respectively.
- f) follows from a) and Proposition 2.4.1. ■

PROPOSITION 2.4.10 Let Ω be a locally compact spaces, $\omega \in \Omega$, Ω' a compact space, and

$$\vartheta : \Omega' \times [0, 1] \longrightarrow \Omega$$

a continuous map such that $\vartheta(\omega', 0) = \omega$ for all $\omega' \in \Omega'$. Then the map

$$\mathcal{C}_0(\Omega \setminus \{\omega\}, F) \longrightarrow \mathcal{C}(\Omega', F), \quad x \longmapsto x \circ \vartheta(\cdot, 1)$$

is K -null

For every $s \in [0, 1]$ put

$$\psi_s : \mathcal{C}_0(\Omega \setminus \{\omega\}, F) \longrightarrow \mathcal{C}(\Omega', F), \quad x \longmapsto x \circ \vartheta(\cdot, s).$$

Then for every $x \in \mathcal{C}_0(\Omega \setminus \{\omega\}, F)$ the map

$$[0, 1] \longrightarrow \mathcal{C}(\Omega', F), \quad s \longmapsto \psi_s x$$

is continuous and $\psi_0 x = 0$, so the assertion follows from the homotopy (Axiom 1.2.5). ■

PROPOSITION 2.4.11 *Let Ω be a locally compact space, Δ a closed set of Ω , Γ a compact set of Δ , $\omega_0 \in \Gamma$ such that $\mathcal{C}_0(\Delta \setminus \{\omega_0\}, F)$ is K -null, and $\vartheta : \Gamma \times [0, 1] \longrightarrow \Omega$ a continuous map such that $\vartheta(\omega, 1) = \omega$ and $\vartheta(\omega, 0) = \omega_0$ for all $\omega \in \Gamma$. Then*

$$K_i(\mathcal{C}_0(\Omega \setminus \Gamma, F)) \approx K_i(\mathcal{C}_0(\Omega \setminus \{\omega_0\}, F)) \times K_{i+1}(\mathcal{C}_0(\Gamma \setminus \{\omega_0\}, F)).$$

In particular if Γ is finite

$$K_i(\mathcal{C}_0(\Omega \setminus \Gamma, F)) \approx K_i(\mathcal{C}_0(\Omega \setminus \{\omega_0\}, F)) \times K_{i+1}(F)^{\text{Card}\Gamma-1}.$$

We use the notation of the Topological triple (Proposition 2.1.11) and put

$$\Omega_1 := \Omega \setminus \{\omega_0\}, \quad \Omega_2 := \Omega \setminus \Gamma, \quad \Omega_3 := \Omega \setminus \Delta.$$

By Proposition 2.4.10, $\psi_{1,2}$ is K -null and the first assertion follows from the Topological triple (Proposition 2.1.11 b_4). The last assertion follows from the first one and from the Product Theorem (Proposition 2.3.1 a). ■

Chapter 3

Some Selected Locally Compact Spaces

Throughout this chapter we endow $\{0, 1\}$ with a group structure by identifying it with \mathbb{Z}_2 , F denotes an E - C^* -algebra, $i \in \{0, 1\}$, and $n \in \mathbb{N}$.

3.1 Balls

DEFINITION 3.1.1 *We put*

$$\mathbb{B}_n := \{ \alpha \in \mathbb{R}^n \mid \|\alpha\| \leq 1 \} .$$

THEOREM 3.1.2 *Let Γ be a closed set of \mathbb{B}_n , $\omega_0 \in \Gamma$, and $\Gamma' := \Gamma \setminus \{\omega_0\}$.*

a) $\mathbb{B}_n \setminus \{\omega_0\}$ is null-homotopic and so Υ -null, $\mathbb{B}_n \in \Upsilon_1$, and every exact sequence in \mathfrak{M}_E belongs to $(\mathbb{B}_n)_\Upsilon$. We use in the sequel the notation of Proposition 2.3.6 and put there $\Omega := \mathbb{B}_n$.

b) $K_i(\mathcal{C}_0(\mathbb{B}_n \setminus \Gamma, F)) \approx K_{i+1}(\mathcal{C}_0(\Gamma', F))$.

c)

$$\begin{aligned} 0 \longrightarrow K_i(\mathcal{C}_0(\mathbb{B}_n, F)) &\xrightarrow{K_i(\psi_{1,3})} K_i(\mathcal{C}_0(\Gamma, F)) \xleftarrow{\frac{\delta_{1,3,i}}{\Phi_i}} \\ &\xleftarrow{\frac{\delta_{1,3,i}}{\Phi_i}} K_{i+1}(\mathcal{C}_0(\mathbb{B}_n \setminus \Gamma, F)) \longrightarrow 0 \end{aligned}$$

is a split exact sequence, and the maps

$$K_i(\mathcal{C}_0(\mathbb{B}_n, F)) \times K_{i+1}(\mathcal{C}_0(\mathbb{B}_n \setminus \Gamma, F)) \longrightarrow K_i(\mathcal{C}_0(\Gamma, F)),$$

$$(a, b) \longmapsto K_i(\psi_{1,3})a + \Phi_i b,$$

$$\delta_{2,3,i} : K_i(\mathcal{C}_0(\Gamma', F)) \longrightarrow K_{i+1}(\mathcal{C}_0(\mathbb{B}_n \setminus \Gamma, F))$$

are group isomorphisms.

d) If $\mathbb{B}_n \setminus \Gamma \in \Upsilon$ or $\Gamma' \in \Upsilon$ then with the notation of Corollary 2.1.9

$$\delta_i : K_i(\mathcal{C}_0(\Gamma', F)) \longrightarrow K_{i+1}(\mathcal{C}_0(\mathbb{B}_n \setminus \{\omega_0\}, F))$$

is a group isomorphism and

$$\mathbb{B}_n \setminus \Gamma, \Gamma' \in \Upsilon, \quad p(\mathbb{B}_n \setminus \Gamma) = q(\Gamma'), \quad q(\mathbb{B}_n \setminus \Gamma) = p(\Gamma'),$$

$$\Phi_{i,(\mathbb{B}_n \setminus \Gamma),F} = \delta_{i+1} \circ \Phi_{(i+1),\Gamma',F} .$$

e) Assume Γ finite.

$e_1)$

$$(\delta_{2,3,i})^{-1} : K_{i+1}(\mathcal{C}_0(\mathbb{B}_n \setminus \Gamma, F)) \longrightarrow K_i(F)^{\Gamma'}$$

is a group isomorphism.

$e_2)$

$$K_i(\psi_{1,3}) : K_i(\mathcal{C}(\mathbb{B}_n, F)) \approx K_i(F) \longrightarrow K_i(\mathcal{C}(\Gamma, F)) \approx K_i(F)^{\Gamma},$$

$$a \longmapsto (a)_{\omega \in \Gamma},$$

and, if we identify $K_{i+1}(\mathcal{C}_0(\mathbb{B}_n \setminus \Gamma, F))$ with $K_i(F)^{\Gamma'}$ using the above group isomorphism $(\delta_{2,3,i})^{-1}$, then

$$\delta_{1,3,i} : K_i(\mathcal{C}(\Gamma, F)) \longrightarrow K_i(F)^{Card \Gamma'}, \quad (a_{\omega})_{\omega \in \Gamma} \longmapsto (a_{\omega} - a_{\omega_0})_{\omega \in \Gamma'}.$$

$e_3)$

$$\mathbb{B}_n \setminus \Gamma \in \Upsilon, \quad p(\mathbb{B}_n \setminus \Gamma) = 0, \quad q(\mathbb{B}_n \setminus \Gamma) = Card \Gamma',$$

$$\Phi_{i,(\mathbb{B}_n \setminus \Gamma),F} = \delta_{2,3,(i+1)} \circ \Phi_{(i+1),\Gamma',F},$$

a) Since \mathbb{B}_n is homeomorphic to $[0, 1]^n$, it follows from Corollary 2.4.2 that $\mathcal{C}_0(\Omega \setminus \{\omega_0\}, \mathbb{C})$ is null-homotopic and $\mathbb{B}_n \in \Upsilon_1$. By Proposition 1.5.4 d), $\mathbb{B}_n \setminus \{\omega_0\}$ is Υ -null and by Proposition 1.6.6, every exact sequence in \mathfrak{M}_E belongs to $(\mathbb{B}_n)_{\Upsilon}$.

b), c), d), $e_1)$, and $e_3)$ follow from a) and Proposition 2.3.6.

$e_2)$ follows from a) and Proposition 2.4.8 $e_3)$, $e_4)$. ■

Remark. By b), $K_i(\mathcal{C}_0(\mathbb{B}_n \setminus \Gamma, F))$ depends only on $K_{i+1}(\mathcal{C}_0(\Gamma', F))$ and not on n or on the embedding of Γ in \mathbb{B}_n .

COROLLARY 3.1.3 *Let $(\Gamma_j)_{j \in J}$ be a finite family of pairwise disjoint closed sets of \mathbb{B}_n , $J \neq \emptyset$, and for every $j \in J$ let $\omega_j \in \Gamma_j$ such that $\mathcal{C}_0(\Gamma_j \setminus \{\omega_j\}, F)$ is K -null. Then*

$$\begin{aligned} K_i \left(\mathcal{C}_0 \left(\mathbb{B}_n \setminus \bigcup_{j \in J} \Gamma_j, F \right) \right) &\approx \\ &\approx K_i(\mathcal{C}_0(\mathbb{B}_n \setminus \{\omega_j \mid j \in J\}, F)) \approx K_{i+1}(F)^{Card J - 1} \end{aligned}$$

$$\text{Put } \Gamma := \bigcup_{j \in J} (\Gamma_j \setminus \{\omega_j\}),$$

$$\psi : \mathcal{C}_0(\mathbb{B}_n \setminus \{\omega_j \mid j \in J\}, F) \longrightarrow \mathcal{C}_0(\Gamma, F), \quad x \longmapsto x|_\Gamma,$$

and denote by $\varphi : \mathcal{C}_0\left(\mathbb{B}_n \setminus \bigcup_{j \in J} \Gamma_j, F\right) \longrightarrow \mathcal{C}_0(\mathbb{B}_n \setminus \{\omega_j \mid j \in J\}, F)$ the inclusion map.

Then

$$0 \longrightarrow \mathcal{C}_0\left(\mathbb{B}_n \setminus \bigcup_{j \in J} \Gamma_j, F\right) \xrightarrow{\varphi} \mathcal{C}_0(\mathbb{B}_n \setminus \{\omega_j \mid j \in J\}, F) \xrightarrow{\psi} \mathcal{C}_0(\Gamma, F) \longrightarrow 0$$

is an exact sequence in \mathfrak{M}_E . By the Product Theorem (Proposition 2.3.1 c)), $\mathcal{C}_0(\Gamma, F)$ is K-null so by the Topological six-term sequence (Proposition 2.1.8 b)) and Theorem 3.1.2 e_1),

$$\begin{aligned} K_i\left(\mathcal{C}_0\left(\mathbb{B}_n \setminus \bigcup_{j \in J} \Gamma_j, F\right)\right) &\approx \\ &\approx K_i(\mathcal{C}_0(\mathbb{B}_n \setminus \{\omega_j \mid j \in J\}, F)) \approx K_{i+1}(F)^{\text{Card } J - 1}. \end{aligned} \quad \blacksquare$$

COROLLARY 3.1.4 *Let $(k_j)_{j \in J}$ be a finite family in \mathbb{N} and for every $j \in J$ let Γ_j be a nonempty finite subset of \mathbb{B}_{k_j} . If Ω denotes the Alexandroff compactification of the topological sum of the family $(\mathbb{B}_{k_j} \setminus \Gamma_j)_{j \in J}$ then*

$$\Omega \in \Upsilon, \quad p(\Omega) = 1, \quad q(\Omega) = \sum_{j \in J} (\text{Card } \Gamma_j - 1).$$

For every $j \in J$ let $\omega_j \in \Gamma_j$. By Theorem 3.1.2 a), $\mathbb{B}_{k_j} \setminus \{\omega_j\}$ is Υ -null and the assertion follows from Corollary 2.3.5 b). \blacksquare

COROLLARY 3.1.5 *If Ω is a path connected compact space, $\omega \in \Omega$, and $\omega' \in \mathbb{B}_n \times \Omega$ then*

$$K_i(\mathcal{C}_0(\Omega \setminus \{\omega\}, F)) \approx K_i(\mathcal{C}_0(\mathbb{B}_n \times \Omega \setminus \{\omega'\}, F)).$$

By Theorem 3.1.2 a), $\mathcal{C}_0(\mathbb{B}_n \setminus \{\omega_0\}, F)$ is K-null for every $\omega_0 \in \mathbb{B}_n$ and the assertion follows from Corollary 2.4.5. \blacksquare

COROLLARY 3.1.6 *Let Γ be a closed set of \mathbb{B}_n and Ω an open set of \mathbb{B}_n , $\Omega \subset \Gamma$. Then for all $\omega \in \Gamma \setminus \Omega$,*

$$K_i(\mathcal{C}_0((\Gamma \setminus \Omega) \setminus \{\omega\}, F)) \approx K_i(\mathcal{C}_0(\Gamma \setminus \{\omega\}, F)) \times K_{i+1}(\mathcal{C}_0(\Omega, F)) ,$$

$$K_i(\mathcal{C}_0(\Gamma \setminus \Omega, F)) \approx K_i(\mathcal{C}(\Gamma, F)) \times K_{i+1}(\mathcal{C}_0(\Omega, F)) .$$

By Theorem 3.1.2 b),

$$K_i(\mathcal{C}_0(\Gamma \setminus \{\omega\}, F)) \approx K_{i+1}(\mathcal{C}_0(\mathbb{B}_n \setminus \Gamma, F)) ,$$

$$K_i(\mathcal{C}_0((\Gamma \setminus \Omega) \setminus \{\omega\}, F)) \approx K_{i+1}(\mathcal{C}_0(\mathbb{B}_n \setminus (\Gamma \setminus \Omega), F))$$

and by the Product Theorem (Proposition 2.3.1a)),

$$K_{i+1}(\mathcal{C}_0(\mathbb{B}_n \setminus (\Gamma \setminus \Omega), F)) \approx K_{i+1}(\mathcal{C}_0(\mathbb{B}_n \setminus \Gamma, F)) \times K_{i+1}(\mathcal{C}_0(\Omega, F)) ,$$

so

$$K_i(\mathcal{C}_0((\Gamma \setminus \Omega) \setminus \{\omega\}, F)) \approx K_i(\mathcal{C}_0(\Gamma \setminus \{\omega\}, F)) \times K_{i+1}(\mathcal{C}_0(\Omega, F)) .$$

The last relation follows from the Alexandroff K-theorem (Proposition 2.2.1 a)). ■

COROLLARY 3.1.7 *If Ω is an open set of \mathbb{B}_n , $\Omega \neq \mathbb{B}_n$, and Γ a compact set of Ω then*

$$K_i(\mathcal{C}_0(\Omega \setminus \Gamma, F)) \approx K_i(\mathcal{C}_0(\Omega, F)) \times K_{i+1}(\mathcal{C}(\Gamma, F)) .$$

Let $\omega \in \mathbb{B}_n \setminus \Omega$. By Theorem 3.1.2 b),

$$K_i(\mathcal{C}_0(\Omega, F)) \approx K_{i+1}(\mathcal{C}_0((\mathbb{B}_n \setminus \Omega) \setminus \{\omega\}, F)) ,$$

$$K_i(\mathcal{C}_0(\Omega \setminus \Gamma, F)) \approx K_{i+1}(\mathcal{C}_0(((\mathbb{B}_n \setminus \Omega) \setminus \{\omega\}) \cup \Gamma, F)) .$$

By the Product Theorem (Proposition 2.3.1 a)),

$$K_{i+1}(\mathcal{C}_0(((\mathbb{B}_n \setminus \Omega) \setminus \{\omega\}) \cup \Gamma, F)) \approx$$

$$\approx K_{i+1}(\mathcal{C}_0((\mathbb{B}_n \setminus \Omega) \setminus \{\omega\}, F)) \times K_{i+1}(\mathcal{C}(\Gamma, F)) ,$$

so

$$K_i(\mathcal{C}_0(\Omega \setminus \Gamma, F)) \approx K_i(\mathcal{C}_0(\Omega, F)) \times K_{i+1}(\mathcal{C}(\Gamma, F)) .$$

■

3.2 Euclidean Spaces and Spheres

DEFINITION 3.2.1 We put

$$\mathbf{S}_{n-1} := \{ \alpha \in \mathbb{R}^n \mid \|\alpha\| = 1 \}, \quad \mathbf{T} := \mathbf{S}_1.$$

THEOREM 3.2.2

$$a) \quad \mathbb{R}^n \in \Upsilon, \quad p(\mathbb{R}^n) = \frac{1+(-1)^n}{2}, \quad q(\mathbb{R}^n) = \frac{1-(-1)^n}{2},$$

$$\mathbb{R}_\Upsilon \subset (\mathbb{R}^n)_\Upsilon, \quad K_i(\mathcal{C}_0(\mathbb{R}^n, F)) \approx K_{i+n}(F).$$

$$b) \quad \mathbf{S}_n \in \Upsilon, \quad p(\mathbf{S}_n) = \frac{3+(-1)^n}{2}, \quad q(\mathbf{S}_n) = \frac{1-(-1)^n}{2}, \quad \mathbb{R}_\Upsilon \subset (\mathbf{S}_n)_\Upsilon,$$

$$K_i(\mathcal{C}(\mathbf{S}_n, F)) \approx$$

$$\approx \begin{cases} K_i(F)^2 & \text{if } n \text{ is even} \\ K_i(F) \times K_{i+1}(F) & \text{if } n \text{ is odd} \end{cases} = K_i(F) \times K_{i+n}(F),$$

and the map

$$K_i(F) \times K_{i+n}(F) \longrightarrow K_i(\mathcal{C}(\mathbf{S}_n, F)), \quad (a, b) \longmapsto K_i(\lambda)a + K_{i+n}(\varphi)b$$

is a group isomorphism, where $\varphi : \mathcal{C}_0(\mathbb{R}^n, F) \approx K_{i+n}(F) \longrightarrow \mathcal{C}(\mathbf{S}_n, F)$ denotes the inclusion map and

$$\lambda : F \longrightarrow \mathcal{C}(\mathbf{S}_n, F), \quad x \longmapsto x1_{\mathcal{C}(\mathbb{S}_n, \mathbb{C})}.$$

c) Let Γ be a closed set of \mathbb{R}^n , $\Gamma \neq \mathbb{R}^n$.

c₁) The map

$$\mathcal{C}_0(\mathbb{R}^n, F) \longrightarrow \mathcal{C}_0(\Gamma, F), \quad x \longmapsto x|_\Gamma$$

is K -null.

c₂) If Γ is compact then

$$K_i(\mathcal{C}_0(\mathbb{R}^n \setminus \Gamma, F)) \approx K_{i+n}(F) \times K_{i+1}(\mathcal{C}(\Gamma, F)).$$

If in addition $\Gamma \in \Upsilon$ then $\mathbb{R}^n \setminus \Gamma \in \Upsilon$, and

$$p(\mathbb{R}^n \setminus \Gamma) = \begin{cases} q(\Gamma) + 1 & \text{if } n \text{ is even} \\ q(\Gamma) & \text{if } n \text{ is odd} \end{cases},$$

$$q(\mathbb{R}^n \setminus \Gamma) = \begin{cases} p(\Gamma) & \text{if } n \text{ is even} \\ p(\Gamma) + 1 & \text{if } n \text{ is odd} \end{cases}.$$

d) If Γ is finite then $\mathbb{R}^n \setminus \Gamma \in \Upsilon$, and

$$p(\mathbb{R}^n \setminus \Gamma) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases},$$

$$q(\mathbb{R}^n \setminus \Gamma) = \begin{cases} \text{Card}\Gamma & \text{if } n \text{ is even} \\ \text{Card}\Gamma + 1 & \text{if } n \text{ is odd} \end{cases}.$$

e) Let Γ be a closed set of \mathbf{S}_n , $\Gamma \neq \mathbf{S}_n$, $\omega \in \Gamma$, and $\Gamma' := \Gamma \setminus \{\omega\}$.

$$e_1) K_i(\mathcal{C}_0(\mathbf{S}_n \setminus \Gamma, F)) \approx K_{i+n}(F) \times K_{i+1}(\mathcal{C}_0(\Gamma \setminus \{\omega\}, F)).$$

e₂) If $\Gamma' \in \Upsilon$ then $\mathbf{S}_n \setminus \Gamma \in \Upsilon$, and

$$p(\mathbf{S}_n \setminus \Gamma) = \begin{cases} q(\Gamma') + 1 & \text{if } n \text{ is even} \\ q(\Gamma') & \text{if } n \text{ is odd} \end{cases},$$

$$q(\mathbf{S}_n \setminus \Gamma) = \begin{cases} p(\Gamma') & \text{if } n \text{ is even} \\ p(\Gamma') + 1 & \text{if } n \text{ is odd} \end{cases}.$$

e₃) If Γ is finite, then $\mathbf{S}_n \setminus \Gamma \in \Upsilon$, and

$$p(\mathbf{S}_n \setminus \Gamma) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases},$$

$$q(\mathbf{S}_n \setminus \Gamma) = \begin{cases} \text{Card}\Gamma' & \text{if } n \text{ is even} \\ \text{Card}\Gamma & \text{if } n \text{ is odd} \end{cases}.$$

f) If $m \in \mathbb{N}$, $m < n$, then

$$K_i(\mathcal{C}_0(\mathbf{S}_n \setminus \mathbf{S}_m, F)) \approx K_i(\mathcal{C}_0(\mathbb{R}^n \setminus \mathbb{R}^m, F)) \approx K_i(F) \times K_{i+n-m+1}(F).$$

g) For $m \in \mathbb{N}$, $m < n$,

$$K_i(\mathcal{C}_0(\mathbb{B}_n \setminus \mathbf{S}_m, F)) \approx K_{i+m+1}(F).$$

a) Since \mathbb{R} is homeomorphic to $]0, 1[= \mathbb{I}B_1 \setminus \{-1, 1\}$ we get

$$\mathbb{R} \in \Upsilon, \quad p(\mathbb{R}) = 0, \quad q(\mathbb{R}) = 1$$

from Theorem 3.1.2 e_3) and the assertion follows from Corollary 1.5.12.

b) Since \mathbf{S}_n is homeomorphic to the Alexandroff compactification of \mathbb{R}^n , b) follows from a) and the Alexandroff K-theorem (Theorem 2.2.1 a), b)).

c1) We may assume $0 \in \mathbb{R}^n \setminus \Gamma$. Put

$$\vartheta : \Gamma \times]0, 1] \longrightarrow \mathbb{R}^n, \quad (\omega, s) \longmapsto \frac{1}{s} \omega$$

and for every $s \in [0, 1]$

$$\psi_s : \mathcal{C}_0(\mathbb{R}^n, F) \longrightarrow \mathcal{C}_0(\Gamma, F), \quad x \longmapsto \begin{cases} x \circ \vartheta(\cdot, s) & \text{if } s \neq 0 \\ 0 & \text{if } s = 0 \end{cases}.$$

Then for every $x \in \mathcal{C}_0(\mathbb{R}^n, F)$,

$$[0, 1] \longrightarrow \mathcal{C}_0(\Gamma, F), \quad s \longmapsto \psi_s x$$

is continuous, $\psi_1 x = x|_\Gamma$, and $\psi_0 x = 0$. Thus the assertion follows from the homotopy axiom (Axiom 1.2.5).

c2) We identify the homeomorphic spaces $\{\alpha \in \mathbb{R}^n \mid \|\alpha\| < 1\}$ and \mathbb{R}^n , put $\omega := (1, 0, \dots, 0) \in \mathbb{I}B_n$ and

$$\psi : \mathcal{C}_0(\mathbb{I}B_n \setminus \{\omega\}, F) \longrightarrow \mathcal{C}_0((\mathbf{S}_{n-1} \setminus \{\omega\}) \cup \Gamma, F), \quad x \longmapsto x|((\mathbf{S}_{n-1} \setminus \{\omega\}) \cup \Gamma),$$

and denote by $\varphi : \mathcal{C}_0(\mathbb{R}^n \setminus \Gamma, F) \longrightarrow \mathcal{C}_0(\mathbb{I}B_n \setminus \{\omega\}, F)$ the inclusion map and by δ_i the index maps associated to the exact sequence in \mathfrak{M}_E

$$0 \longrightarrow \mathcal{C}_0(\mathbb{R}^n \setminus \Gamma, F) \xrightarrow{\varphi} \mathcal{C}_0(\mathbb{I}B_n \setminus \{\omega\}, F) \xrightarrow{\psi} \mathcal{C}_0((\mathbf{S}_{n-1} \setminus \{\omega\}) \cup \Gamma, F) \longrightarrow 0.$$

By Theorem 3.1.2 a), $\mathcal{C}_0(\mathbb{I}B_n \setminus \{\omega\}, F)$ is K-null so by the Topological six-term sequence (Proposition 2.1.8 c)), the map

$$\delta_{i+1} : K_{i+1}(\mathcal{C}_0((\mathbf{S}_{n-1} \setminus \{\omega\}) \cup \Gamma, F)) \longrightarrow K_i(\mathcal{C}_0(\mathbb{R}^n \setminus \Gamma, F))$$

is a group isomorphism. By the Product Theorem (Proposition 2.3.1 a), b)),

$$K_{i+1}(\mathcal{C}_0((\mathbf{S}_{n-1} \setminus \{\omega\}) \cup \Gamma, F)) \approx K_{i+1}(\mathcal{C}_0(\mathbf{S}_{n-1} \setminus \{\omega\}, F)) \times K_{i+1}(\mathcal{C}_0(\Gamma, F)),$$

and $\Gamma \in \Upsilon$ implies $\mathbb{R}^n \setminus \Gamma \in \Upsilon$. By a), $K_{i+1}(\mathcal{C}_0(\mathbf{S}_{n-1} \setminus \{\omega\}, F)) \approx K_{i+n}(F)$ so

$$K_i(\mathcal{C}_0(\mathbb{R}^n \setminus \Gamma, F)) \approx K_{i+n}(F) \times K_{i+1}(\mathcal{C}(\Gamma, F))$$

as well as the last assertions.

d) follows from c) and the Product Theorem (Proposition 2.3.1 a),b)).

e) $\mathbf{S}_n \setminus \Gamma$ is homeomorphic to $\mathbb{R}^n \setminus (\Gamma \setminus \{\omega\})$ and the assertion follows from c) and d).

f) Step 1

$$K_i(\mathcal{C}_0(\mathbf{S}_n \setminus \mathbf{S}_m, F)) \approx K_i(\mathcal{C}_0(\mathbb{R}^n \setminus \mathbb{R}^m, F))$$

Let $\omega \in \mathbf{S}_m$. Then $\mathbf{S}_n \setminus \mathbf{S}_m = (\mathbf{S}_n \setminus \{\omega\}) \setminus (\mathbf{S}_m \setminus \{\omega\})$. Since $(\mathbf{S}_n \setminus \{\omega\}) \setminus (\mathbf{S}_m \setminus \{\omega\})$ is homeomorphic to $\mathbb{R}^n \setminus \mathbb{R}^m$ we get

$$K_i(\mathcal{C}_0(\mathbf{S}_n \setminus \mathbf{S}_m, F)) \approx K_i(\mathcal{C}_0(\mathbb{R}^n \setminus \mathbb{R}^m, F)) .$$

Step 2

$$K_i(\mathcal{C}_0(\mathbb{R}^n \setminus \mathbb{R}^m, F)) \approx K_i(F) \times K_{i+n-m+1}(F)$$

We identify $\mathbb{R}^n \setminus \mathbb{R}^m$ with $\left\{ \alpha \in \mathbb{I}\mathbb{B}_n \mid \|\alpha\| < 1, \sum_{j=m+1}^n \alpha_j^2 \neq 0 \right\}$, put

$$\psi : \mathbb{I}\mathbb{B}_n \setminus \mathbb{I}\mathbb{B}_m \longrightarrow \mathbf{S}_{n-1} \setminus \mathbf{S}_{m-1}, \quad x \longmapsto x|(\mathbf{S}_{n-1} \setminus \mathbf{S}_{m-1}),$$

and denote by $\varphi : \mathbb{R}^n \setminus \mathbb{R}^m \longrightarrow \mathbb{I}\mathbb{B}_n \setminus \mathbb{I}\mathbb{B}_m$ the inclusion map and by δ_i the index maps associated to the exact sequence in \mathfrak{M}_E

$$0 \longrightarrow \mathcal{C}_0(\mathbb{R}^n \setminus \mathbb{R}^m, F) \xrightarrow{\varphi} \mathcal{C}_0(\mathbb{I}\mathbb{B}_n \setminus \mathbb{I}\mathbb{B}_m, F) \xrightarrow{\psi} \mathcal{C}_0(\mathbf{S}_{n-1} \setminus \mathbf{S}_{m-1}, F) \longrightarrow 0 .$$

By Proposition 2.4.1, $\mathcal{C}_0(\mathbb{I}\mathbb{B}_n \setminus \mathbb{I}\mathbb{B}_m, F)$ is K-null so by the Topological six-term sequence (Proposition 2.1.8 c)) and Step 1,

$$\begin{aligned} K_i(\mathcal{C}_0(\mathbb{R}^n \setminus \mathbb{R}^m, F)) &\approx K_{i+1}(\mathcal{C}_0(\mathbf{S}_{n-1} \setminus \mathbf{S}_{m-1}, F)) \approx \\ &\approx K_{i+1}(\mathcal{C}_0(\mathbb{R}^{n-1} \setminus \mathbb{R}^{m-1}, F)) . \end{aligned}$$

For $m = 1$, by e_1),

$$K_i(\mathcal{C}_0(\mathbb{R}^n \setminus \mathbb{R}, F)) \approx K_{i+1}(\mathcal{C}_0(\mathbf{S}_{n-1} \setminus \mathbf{S}_0, F)) \approx K_{i+n}(F) \times K_i(F) .$$

By induction and by the above,

$$\begin{aligned} K_i(\mathcal{C}_0(\mathbb{R}^n \setminus \mathbb{R}^m, F)) &\approx K_{i+n-m+1}(\mathcal{C}_0(\mathbb{R}^{n-m+1} \setminus \mathbb{R}, F)) \approx \\ &\approx K_{i+n-m+1}(F) \times K_i(F) . \end{aligned}$$

g) Let $\omega \in \mathbf{S}_m$. Since $\mathbf{S}_m \setminus \{\omega\}$ is homeomorph to \mathbb{R}^m , by a),

$$K_i(\mathcal{C}_0(\mathbf{S}_m \setminus \{\omega\}, F)) \approx K_{i+m}(F) .$$

By Theorem 3.1.2 b),

$$K_i(\mathcal{C}_0(\mathbb{B}_n \setminus \mathbf{S}_m, F)) \approx K_{i+1}(\mathcal{C}_0(\mathbf{S}_m \setminus \{\omega\}, F)) \approx K_{i+1+m}(F) . \quad \blacksquare$$

EXAMPLE 3.2.3 Put

$$\begin{aligned} \Omega_1 &:= \mathbf{S}_1 \cup \left\{ re^{\frac{2\pi ij}{n}} \mid r \in [0, 1], j \in \mathbb{N}_n \right\} , \\ \Omega_2 &:= \mathbf{S}_2 \cup \{ \alpha \in \mathbb{B}_3 \mid \alpha_3 = 0 \} \cup \{ \alpha \in \mathbb{B}_3 \mid \alpha_1 = \alpha_2 = 0 \} , \\ \Omega_3 &:= \mathbf{S}_{n-1} \cup \left(\bigcup_{j \in \mathbb{N}_n} \{ \alpha \in \mathbb{B}_n \mid \alpha_j = 0 \} \right) . \end{aligned}$$

- a) $K_i(\mathcal{C}(\Omega_1, F)) = K_i(F) \times K_{i+1}(F)^n$.
- b) $K_i(\mathcal{C}(\Omega_2, F)) \approx K_i(F)^3 \times K_{i+1}(F)^2$.
- c) $K_i(\mathcal{C}(\Omega_3, F)) = K_i(F) \times K_{i+n+1}(F)^{2^n}$.

a) By Theorem 3.2.2 b) and the Product Theorem (Proposition 2.3.1 a)),

$$K_i(\mathcal{C}_0(\mathbb{B}_2 \setminus \Omega_1, F)) \approx K_i(F)^n$$

and by Theorem 3.1.2 a),b),c),

$$K_i(\mathcal{C}(\Omega_1, F)) \approx K_i(\mathcal{C}(\mathbb{B}_2, F)) \times K_{i+1}(\mathcal{C}_0(\mathbb{B}_2 \setminus \Omega_1, F)) \approx K_i(F) \times K_{i+1}(F)^n .$$

b) By Theorem 3.2.2 a),b),

$$\mathbb{R}^2, \mathbf{S}_1 \in \Upsilon, \quad p(\mathbb{R}^2) = 1, \quad q(\mathbb{R}^2) = 0, \quad p(\mathbf{S}_1) = 1, \quad q(\mathbf{S}_1) = 1 ,$$

so by Corollary 1.5.11 d_1),

$$K_i(\mathcal{C}_0(\mathbb{R}^2 \times \mathbf{S}_1, F)) \approx K_i(F) \times K_{i+1}(F) .$$

Since $\mathbb{I}\mathbb{B}_3 \setminus \Omega_2$ is homeomorphic to the topological sum of two copies of $\mathbb{R}^2 \times \mathbf{S}_1$ we get by the Product Theorem (Proposition 2.3.1 a))

$$K_i(\mathcal{C}_0(\mathbb{I}\mathbb{B}_3 \setminus \Omega_2, F)) \approx K_i(F)^2 \times K_{i+1}(F)^2 .$$

By Theorem 3.1.2 a),b),c),

$$K_i(\mathcal{C}(\Omega_2, F)) \approx K_i(\mathcal{C}(\mathbb{I}\mathbb{B}_3, F)) \times K_{i+1}(\mathcal{C}_0(\mathbb{I}\mathbb{B}_3 \setminus \Omega_2, F)) \approx K_i(F)^3 \times K_{i+1}(F)^2 .$$

c) By Theorem 3.2.2 a), $K_i(\mathcal{C}_0(\mathbb{R}^n, F)) \approx K_{i+n}(F)$. Since $\mathbb{I}\mathbb{B}_n \setminus \Omega_3$ is homeomorphic to the topological sum of 2^n copies of \mathbb{R}^n , we get by the Product Theorem (Proposition 2.3.1 a)) $K_i(\mathcal{C}_0(\mathbb{I}\mathbb{B}_n \setminus \Omega_3, F)) \approx K_{i+n}(F)^{2^n}$. By Theorem 3.1.2 a),b),c),

$$\begin{aligned} K_i(\mathcal{C}(\Omega_3, F)) &\approx K_i(\mathcal{C}(\mathbb{I}\mathbb{B}_n, F)) \times K_{i+1}(\mathcal{C}_0(\mathbb{I}\mathbb{B}_n \setminus \Omega_3, F)) \approx \\ &\approx K_i(F) \times K_{i+n+1}(F)^{2^n} . \end{aligned} \quad \blacksquare$$

Remark. The above a) and b) will be generalized in Example 3.5.11 b) and c), respectively.

COROLLARY 3.2.4 *Let $(k_j)_{j \in J}$ be a finite family in \mathbb{N} and*

$$p := \text{Card} \{ j \in J \mid k_j \text{ is even} \} , \quad q := \text{Card} \{ j \in J \mid k_j \text{ is odd} \} .$$

a) *If Ω denotes the Alexandroff compactification of the topological sum of the family $(\mathbb{R}^{k_j})_{j \in J}$ then*

$$\Omega \in \Upsilon, \quad \mathbb{R}_\Upsilon \subset \Omega_\Upsilon, \quad p(\Omega) = p + 1, \quad q(\Omega) = q .$$

b) *For every $j \in J$ let $\omega_j \in \mathbf{S}_{k_j}$ and let Ω' denote the compact space obtained from the topological sum of the family $(\mathbf{S}_{k_j})_{j \in J}$ by identifying all the points of the family $(\omega_j)_{j \in J}$. If $J \neq \emptyset$ then*

$$\Omega' \in \Upsilon, \quad \mathbb{R}_\Upsilon \subset \Omega'_\Upsilon, \quad p(\Omega') = p + 1, \quad q(\Omega') = q .$$

In particular if $k_j = 1$ for all $j \in J$ then $p(\Omega') = 1$, $q(\Omega') = \text{Card } J$.

a) By Theorem 3.2.2 a), $\mathbb{R}^{k_j} \in \Upsilon$, $\mathbb{R}_\Upsilon \subset (\mathbb{R}^{k_j})_\Upsilon$,

$$p(\mathbb{R}^{k_j}) = \begin{cases} 1 & \text{if } k_j \text{ is even} \\ 0 & \text{if } k_j \text{ is odd} \end{cases}, \quad q(\mathbb{R}^{k_j}) = \begin{cases} 0 & \text{if } k_j \text{ is even} \\ 1 & \text{if } k_j \text{ is odd} \end{cases}$$

for every $j \in J$. The assertion follows now from the Product Theorem (Proposition 2.3.1 b)) and from Alexandroff's K-theorem (Proposition 2.2.1 b)).

b) follows from a) since Ω and Ω' are homeomorphic. ■

COROLLARY 3.2.5 *Let $(k_j)_{j \in J}$ be a finite family in \mathbb{N} ,*

$$p := \text{Card} \{ j \in J \mid k_j \text{ is even} \}, \quad q := \text{Card} \{ j \in J \mid k_j \text{ is odd} \},$$

$(\Gamma_j)_{j \in J}$ a pairwise disjoint family of closed sets of \mathbb{B}_n such that Γ_j is homeomorphic to \mathbf{S}_{k_j} for every $j \in J$, and $\Gamma := \bigcup_{j \in J} \Gamma_j$. Then

$$\mathbb{B}_n \setminus \Gamma \in \Upsilon, \quad \mathbb{R}_\Upsilon \subset (\mathbb{B}_n \setminus \Gamma)_\Upsilon, \quad p(\mathbb{B}_n \setminus \Gamma) = q, \quad q(\mathbb{B}_n \setminus \Gamma) = 2p - 1.$$

By Theorem 3.2.2 a),b), for $j \in J$,

$$\begin{aligned} \mathbb{R}^{k_j}, \mathbf{S}_{k_j} &\in \Upsilon, \quad \mathbb{R}_\Upsilon \subset (\mathbb{R}^{k_j})_\Upsilon \cap (\mathbf{S}_{k_j})_\Upsilon, \\ p(\mathbb{R}^{k_j}) &= \frac{1 + (-1)^{k_j}}{2}, \quad q(\mathbb{R}^{k_j}) = \frac{1 - (-1)^{k_j}}{2}, \\ p(\mathbf{S}_{k_j}) &= \frac{3 + (-1)^{k_j}}{2}, \quad q(\mathbf{S}_{k_j}) = \frac{1 - (-1)^{k_j}}{2}. \end{aligned}$$

Let $\omega \in \Gamma$ and $\Gamma' := \Gamma \setminus \{\omega\}$. By the Product Theorem (Proposition 2.3.1 b)),

$$\Gamma' \in \Upsilon, \quad \mathbb{R}_\Upsilon \subset \Gamma'_\Upsilon, \quad p(\Gamma') = 2p - 1, \quad q(\Gamma') = q,$$

so by Theorem 3.1.2 d),

$$\mathbb{B}_n \setminus \Gamma \in \Upsilon, \quad \mathbb{R}_\Upsilon \subset (\mathbb{B}_n \setminus \Gamma)_\Upsilon, \quad p(\mathbb{B}_n \setminus \Gamma) = q, \quad q(\mathbb{B}_n \setminus \Gamma) = 2p - 1. \quad \blacksquare$$

COROLLARY 3.2.6 *If Ω is a connected closed set of \mathbb{B}_2 possessing a triangulation with r_0 vertices, r_1 chords, and r_2 triangles then*

$$K_i(\mathcal{C}(\Omega, F)) \approx K_i(F) \times K_{i+1}(F)^{1-r_0+r_1-r_2}.$$

Sketch of a proof. If Ω has k holes then $r_0 - r_1 + r_2 + k = 1$. By Theorem 3.1.2 c),

$$K_i(\mathcal{C}(\Omega, F)) \approx K_i(F) \times K_{i+1}(\mathcal{C}_0(\mathbb{B}_2 \setminus \Omega, F)) .$$

By Theorem 3.2.2 a) and the Product Theorem (Proposition 2.3.1 a)),

$$K_i(\mathcal{C}_0(\mathbb{B}_2 \setminus \Omega, F)) \approx K_i(F)^k$$

so

$$K_i(\mathcal{C}(\Omega, F)) \approx K_i(F) \times K_{i+1}(F)^{1-r_0+r_1-r_2} . \quad \blacksquare$$

COROLLARY 3.2.7 *We identify the homeomorphic spaces \mathbb{R}^n and*

$$\{ \alpha \in \mathbb{R}^n \mid \|\alpha\| < 1 \} .$$

Let Γ be a finite subset of \mathbb{R}^n , Δ a subset of Γ , $\omega \in \Delta$, $\Gamma' := \Gamma \setminus \{\omega\}$, $\Delta' := \Delta \setminus \{\omega\}$. We use the notation of the Topological triple (Proposition 2.1.11) and put

$$\Omega_1 := \mathbb{B}_n \setminus \{\omega\}, \quad \Omega_2 := \mathbb{R}^n \setminus \Delta, \quad \Omega_3 := \mathbb{R}^n \setminus \Gamma .$$

a) $\delta_{1,2,i}$ and $\delta_{1,3,i}$ are group isomorphisms.

b) $\psi_{2,3}$ is K -null.

c) If we put $\Phi_i := \delta_{1,3,(i+1)} \circ K_{i+1}(\varphi') \circ (\delta_{1,2,(i+1)})^{-1}$ then

$$\begin{aligned} 0 \longrightarrow K_{i+1}(\mathcal{C}(\Gamma \setminus \Delta, F)) &\xrightarrow{\delta_{2,3,(i+1)}} K_i(\mathcal{C}_0(\mathbb{R}^n \setminus \Gamma, F)) \xrightarrow[\Phi_i]{K_i(\varphi_{2,3})} \\ &\xrightarrow[\Phi_i]{K_i(\varphi_{2,3})} K_i(\mathcal{C}_0(\mathbb{R}^n \setminus \Delta, F)) \longrightarrow 0 \end{aligned}$$

is a split exact sequence and the map

$$K_{i+1}(\mathcal{C}(\Gamma \setminus \Delta, F)) \times K_i(\mathcal{C}_0(\mathbb{R}^n \setminus \Delta, F)) \longrightarrow K_i(\mathcal{C}_0(\mathbb{R}^n \setminus \Gamma, F)),$$

$$(a, b) \longmapsto \delta_{2,3,(i+1)}a + \Phi_ib$$

is a group isomorphism.

By Theorem 3.1.2 a), $\mathcal{C}_0(\Omega_1, F)$ is K-null and by Proposition 2.4.10, $\psi_{2,3}$ is K-null. By the Product Theorem (Proposition 2.3.1 a)),

$$K_i(\psi \circ \varphi') = id_{K_i(\mathcal{C}_0(\Omega_1 \setminus \Omega_2, F))}$$

and a) and c) follow from the Topological triple (Proposition 2.1.11 c)). ■

COROLLARY 3.2.8 *Let $\omega \in \mathbf{S}_{n-1}$. We use the notation of the Topological triple (Proposition 2.1.11) and put*

$$\Omega_1 := \mathbb{B}_n, \quad \Omega_2 := \mathbb{B}_n \setminus \{\omega\}, \quad \Omega_3 := \mathbb{B}_n \setminus \mathbf{S}_{n-1}.$$

a) $\varphi_{1,3}$ is K-null.

b) $\delta_{2,3,i} : K_i(\mathcal{C}_0(\mathbf{S}_{n-1} \setminus \{\omega\}, F)) \longrightarrow K_{i+1}(\mathcal{C}_0(\mathbb{B}_n \setminus \mathbf{S}_{n-1}, F))$ is a group isomorphism.

c) If we put $\Phi_i := K_i(\varphi) \circ (\delta_{2,3,i})^{-1}$ then

$$\begin{aligned} 0 \longrightarrow K_i(\mathcal{C}(\mathbb{B}_n, F)) &\xrightarrow{K_i(\psi_{1,3})} K_i(\mathcal{C}(\mathbf{S}_{n-1}, F)) \xrightarrow[\Phi_i]{\delta_{1,3,i}} \\ &\xrightarrow[\Phi_i]{\delta_{1,3,i}} K_{i+1}(\mathcal{C}_0(\mathbb{B}_n \setminus \mathbf{S}_{n-1}, F)) \longrightarrow 0 \end{aligned}$$

is a split exact sequence and the map

$$K_i(\mathcal{C}(\mathbb{B}_n, F)) \times K_{i+1}(\mathcal{C}_0(\mathbb{B}_n \setminus \mathbf{S}_{n-1}, F)) \longrightarrow K_i(\mathcal{C}(\mathbf{S}_{n-1}, F)),$$

$$(a, b) \longmapsto K_i(\psi_{1,3})a + \Phi_i b$$

is a group isomorphism.

d) Let $\phi : G \longrightarrow H$ be a morphism in \mathfrak{M}_E and put

$$\phi_{\mathbb{B}} : \mathcal{C}(\mathbb{B}_n, G) \longrightarrow \mathcal{C}(\mathbb{B}_n, H), \quad x \longmapsto \phi \circ x,$$

$$\phi_{\mathbb{S}} : \mathcal{C}(\mathbf{S}_{n-1}, G) \longrightarrow \mathcal{C}(\mathbf{S}_{n-1}, H), \quad x \longmapsto \phi \circ x,$$

$$\phi_{\mathbb{B}\mathbb{S}} : \mathcal{C}_0(\mathbb{B}_n \setminus \mathbf{S}_{n-1}, G) \longrightarrow \mathcal{C}_0(\mathbb{B}_n \setminus \mathbf{S}_{n-1}, H), \quad x \longmapsto \phi \circ x.$$

If we identify $K_i(\mathcal{C}(\mathbf{S}_{n-1}, F))$ with

$$K_i(\mathcal{C}(\mathbb{B}_n, F)) \times K_{i+1}(\mathcal{C}_0(\mathbb{B}_n \setminus \mathbf{S}_{n-1}, F))$$

for $F \in \{G, H\}$ using the isomorphism of c) then

$$\begin{aligned} K_i(\phi_{\mathbb{S}}) : K_i(\mathcal{C}(\mathbf{S}_{n-1}, G)) &\longrightarrow K_i(\mathcal{C}(\mathbf{S}_{n-1}, H)), \\ (a, b) &\longmapsto (K_i(\phi_{\mathbb{B}})a, K_{i+1}(\phi_{\mathbb{B}\mathbb{S}})b). \end{aligned}$$

By Theorem 3.1.2 a), $\mathcal{C}_0(\mathbb{B}_n \setminus \{\omega\}, F)$ is K -null and the assertion follows from the Topological triple (Proposition 2.1.11 a)) and Corollary 2.1.12 b). ■

PROPOSITION 3.2.9 *Put*

$$\begin{aligned} \Omega &:= \mathbb{B}_{n+1} \setminus \{ \alpha \in \mathbf{S}_n \mid \alpha_{n+1} = 0 \}, \\ \Omega' &:= \mathbf{S}_n \setminus \{ \alpha \in \mathbf{S}_n \mid \alpha_{n+1} = 0 \}, \\ \psi : \mathcal{C}_0(\Omega, F) &\longrightarrow \mathcal{C}_0(\Omega', F), \quad x \longmapsto x|_{\Omega'} \end{aligned}$$

and denote by

$$\varphi : \mathcal{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F) \longrightarrow \mathcal{C}_0(\Omega, F)$$

the inclusion map and by δ_i the index maps associated to the exact sequence in \mathfrak{M}_E

$$0 \longrightarrow \mathcal{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F) \xrightarrow{\varphi} \mathcal{C}_0(\Omega, F) \xrightarrow{\psi} \mathcal{C}_0(\Omega', F) \longrightarrow 0.$$

a)

$$\begin{aligned} K_i(\mathcal{C}_0(\Omega, F)) &\approx K_{i+n}(F), \quad K_i(\mathcal{C}_0(\Omega', F)) \approx K_{i+n}(F)^2, \\ K_{i+1}(\mathcal{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F)) &\approx K_{i+n}(F). \end{aligned}$$

b) *If we identify the groups of a) then*

$$\begin{aligned} \delta_i : K_i(\mathcal{C}_0(\Omega', F)) &\longrightarrow K_{i+1}(\mathcal{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F)), \quad (a, b) \longmapsto a + b, \\ 0 \longrightarrow K_i(\mathcal{C}_0(\Omega, \cdot)) &\xrightarrow{K_i(\psi)} K_i(\mathcal{C}_0(\Omega', \cdot)) \xrightarrow{\delta_i} K_{i+1}(\mathcal{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, \cdot)) \longrightarrow 0 \end{aligned}$$

is an exact sequence, and there is a group automorphism $\Phi_i : K_{i+n}(F) \longrightarrow K_{i+n}(F)$ such that

$$K_i(\psi) : K_i(\mathcal{C}_0(\Omega, F)) \longrightarrow K_i(\mathcal{C}_0(\Omega', F)), \quad a \longmapsto (\Phi_i a, -\Phi_i a).$$

c) If

$$\lambda' : \mathcal{C}_0(\Omega, F) \longrightarrow \mathcal{C}(\mathbb{B}_{n+1}, F),$$

$$\lambda'' : \mathcal{C}_0(\Omega', F) \longrightarrow \mathcal{C}(\mathbf{S}_n, F)$$

denote the inclusion maps and if we identify $K_i(\mathcal{C}_0(\Omega', F))$ with $K_{i+n}(F)^2$ using a) and $K_i(\mathcal{C}(\mathbf{S}_n, F))$ with $K_i(F) \times K_{i+n}(F)$ using Theorem 3.2.2 b) then λ' is K -null and

$$K_i(\lambda'') : K_i(\mathcal{C}_0(\Omega', F)) \longrightarrow K_i(\mathcal{C}(\mathbf{S}_n, F)), \quad (a, b) \longmapsto (0, a + b).$$

a) By Theorem 3.2.2 a), $K_i(\mathcal{C}_0(\mathbb{R}^n, F)) \approx K_{i+n}(F)$. Since $\mathbb{B}_{n+1} \setminus \mathbf{S}_n$ is homeomorphic to \mathbb{R}^{n+1} , $K_{i+1}(\mathcal{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F)) \approx K_{i+n}(F)$. Since Ω' is homeomorphic to the topological sum of \mathbb{R}^n and \mathbb{R}^n , $K_i(\mathcal{C}_0(\Omega', F)) \approx K_{i+n}(F)^2$ by the Product Theorem (Proposition 2.3.1 a)). Put

$$\Gamma := \{ \alpha \in \Omega \mid \alpha_{n+1} = 0 \}$$

and for every $s \in]0, 1]$

$$\vartheta_s : \Omega \setminus \Gamma \longrightarrow \Omega \setminus \Gamma, \quad (\alpha_j)_{j \in \mathbb{N}_{n+1}} \longmapsto ((\alpha_j)_{j \in \mathbb{N}_n}, s\alpha_{n+1}).$$

By Proposition 2.4.1, $\mathcal{C}_0(\Omega \setminus \Gamma, F)$ is K -null, so by the Topological six-term sequence (Proposition 2.1.8 a)), $K_i(\mathcal{C}_0(\Omega, F)) \approx K_i(\mathcal{C}_0(\Gamma, F))$. Since Γ is homeomorphic to \mathbb{R}^n , $K_i(\mathcal{C}_0(\Omega, F)) \approx K_{i+n}(F)$ by the above.

b) Put $\omega := (1, 0, \dots, 0) \in \mathbb{B}_{n+1}$,

$$\psi' : \mathcal{C}_0(\mathbb{B}_{n+1} \setminus \{\omega\}, F) \longrightarrow \mathcal{C}_0(\mathbf{S}_n \setminus \{\omega\}, F), \quad x \longmapsto x|(\mathbf{S}_n \setminus \{\omega\}),$$

and denote by

$$\phi' : \mathcal{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F) \longrightarrow \mathcal{C}_0(\mathbb{B}_{n+1} \setminus \{\omega\}, F),$$

$$\phi'' : \mathcal{C}_0(\Omega, F) \longrightarrow \mathcal{C}_0(\mathbb{B}_{n+1} \setminus \{\omega\}, F)$$

$$\phi''' : \mathcal{C}_0(\Omega', F) \longrightarrow \mathcal{C}_0(\mathbf{S}_n \setminus \{\omega\}, F)$$

the inclusion maps and by δ'_i the six-term sequence index maps associated with the exact sequence in \mathfrak{M}_E

$$0 \longrightarrow \mathcal{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F) \xrightarrow{\phi'} \mathcal{C}_0(\mathbb{B}_{n+1} \setminus \{\omega\}, F) \xrightarrow{\psi'} \mathcal{C}_0(\mathbf{S}_n \setminus \{\omega\}, F) \longrightarrow 0.$$

By Theorem 3.1.2 a), $\mathcal{C}_0(\mathbb{B}_{n+1} \setminus \{\omega\}, F)$ is K-null so by the Topological six-term sequence (Proposition 2.1.8 c)),

$$\delta'_i : K_i(\mathcal{C}_0(\mathbf{S}_n \setminus \{\omega\}, F)) \longrightarrow K_{i+1}(\mathcal{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F))$$

is a group isomorphism. By the commutativity of the index maps (Axiom 1.2.8), $\delta_i = \delta'_i \circ K_i(\varphi''')$. Thus if we identify the above groups using δ'_i then δ_i is identified with $K_i(\varphi''')$. By Corollary 2.3.2

$$K_i(\varphi''') : K_i(\mathcal{C}_0(\Omega', F)) \longrightarrow K_i(\mathcal{C}_0(\mathbf{S}_n \setminus \{\omega\}, F)), \quad (a, b) \longmapsto a + b.$$

Since $\mathbf{S}_n \setminus \{\omega\}$ is homeomorphic to \mathbb{R}^n , we get

$$\delta_i : K_i(\mathcal{C}_0(\Omega', F)) \longrightarrow K_{i+1}(\mathcal{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F)), \quad (a, b) \longmapsto a + b.$$

Thus δ_i is surjective and the other assertions follow from the six-term axiom (Axiom 1.2.7).

c) λ' is K-null since it factorizes through null (Theorem 3.1.2 a)). Put $\omega := (1, 0, \dots, 0) \in \mathbb{B}_{n+1}$ and denote by

$$\lambda''' : \mathcal{C}_0(\mathbf{S}_n \setminus \{\omega\}, F) \longrightarrow \mathcal{C}(\mathbf{S}_n, F)$$

the inclusion map. By the proof of b), since $\lambda'' = \lambda''' \circ \varphi'''$,

$$K_i(\lambda'') : K_i(\mathcal{C}(\Omega', F)) \longrightarrow K_i(\mathcal{C}(\mathbf{S}_n, F)), \quad (a, b) \longmapsto (0, a + b)$$

by the Alexandroff K-theorem (Theorem 2.2.1 a)). ■

PROPOSITION 3.2.10 *Let Γ be a closed set of \mathbb{R}^n , $\Gamma \neq \mathbb{R}^n$,*

$$\varphi : \mathcal{C}_0(\mathbb{R}^n \setminus \Gamma, F) \longrightarrow \mathcal{C}_0(\mathbb{R}^n, F)$$

the inclusion map,

$$\psi : \mathcal{C}_0(\mathbb{R}^n, F) \longrightarrow \mathcal{C}_0(\Gamma, F), \quad x \longmapsto x|_{\Gamma},$$

and δ_i the index maps associated to the exact sequence in \mathfrak{M}_E

$$0 \longrightarrow \mathcal{C}_0(\mathbb{R}^n \setminus \Gamma, F) \xrightarrow{\varphi} \mathcal{C}_0(\mathbb{R}^n, F) \xrightarrow{\psi} \mathcal{C}_0(\Gamma, F) \longrightarrow 0.$$

a) ψ is K -null.

b) The sequence

$$0 \longrightarrow K_{i+1}(\mathcal{C}_0(\Gamma, F)) \xrightarrow{\delta_{i+1}} K_i(\mathcal{C}_0(\mathbb{R}^n \setminus \Gamma, F)) \xrightarrow{K_i(\varphi)} \mathcal{C}_0(\Gamma, F) \longrightarrow 0$$

is exact.

c) Let $(\Omega_j)_{j \in J}$ be a finite family of pairwise disjoint open sets of \mathbb{R}^n the union of which is $\mathbb{R}^n \setminus \Gamma$. If there is a $j_0 \in J$ such that $\mathcal{C}_0(\mathbb{R}^n \setminus \Omega_{j_0}, F)$ is K -null then for every clopen set Γ' of Γ

$$K_i(\mathcal{C}_0(\mathbb{R}^n \setminus \Gamma', F)) \approx K_{i+1}(\mathcal{C}_0(\Gamma', F)) \times K_{i+n}(F).$$

a) follows from Proposition 2.4.10.

b) follows from a) and the six-term axiom (Axiom 1.2.7).

c) We use the notation of Proposition 2.3.7. For $\Gamma' = \Gamma$ the assertion follows from Proposition 2.3.7 c₂) and Theorem 3.2.2 b). Let

$$\tilde{\varphi} : \mathcal{C}_0(\mathbb{R}^n \setminus \Gamma', F) \longrightarrow \mathcal{C}_0(\mathbb{R}^n, F),$$

$$\tilde{\tilde{\varphi}} : \mathcal{C}_0(\mathbb{R}^n \setminus \Gamma, F) \longrightarrow \mathcal{C}_0(\mathbb{R}^n \setminus \Gamma', F)$$

be the inclusion maps,

$$\tilde{\psi} : \mathcal{C}_0(\mathbb{R}^n, F) \longrightarrow \mathcal{C}_0(\Gamma', F), \quad x \longmapsto x|_{\Gamma'},$$

$\tilde{\delta}_i$ the index maps associated to the exact sequence in \mathfrak{M}_E

$$0 \longrightarrow \mathcal{C}_0(\mathbb{R}^n \setminus \Gamma', F) \xrightarrow{\tilde{\varphi}} \mathcal{C}_0(\mathbb{R}^n, F) \xrightarrow{\tilde{\psi}} \mathcal{C}_0(\Gamma', F) \longrightarrow 0,$$

and $\tilde{\Phi}_i := K_i(\tilde{\tilde{\varphi}}) \circ \Phi_i$. Since $\varphi = \tilde{\varphi} \circ \tilde{\tilde{\varphi}}$,

$$K_i(\tilde{\varphi}) \circ \tilde{\Phi}_i = K_i(\tilde{\varphi}) \circ K_i(\tilde{\tilde{\varphi}}) \circ \Phi_i = K_i(\varphi) \circ \Phi_i = id_{K_i(\mathcal{C}_0(\mathbb{R}^n, F))}.$$

Thus

$$0 \longrightarrow K_{i+1}(\mathcal{C}_0(\Gamma', F)) \xrightarrow{\tilde{\delta}_{i+1}} K_i(\mathcal{C}_0(\mathbb{R}^n \setminus \Gamma', F)) \xrightarrow[\leftarrow \Phi_i]{K_i(\tilde{\varphi})} K_i(\mathcal{C}_0(\mathbb{R}^n, F)) \longrightarrow 0$$

is a split exact sequence and this implies c). ■

PROPOSITION 3.2.11 *Let Ω, Ω' be compact spaces and $m \in \mathbb{N}$. If Ω is path connected, $\Omega \times \Omega' \subset \mathbb{I}\mathbb{B}_n$, and $\mathbb{I}\mathbb{B}_n \setminus (\Omega \times \Omega')$ is homeomorphic to the topological sum of $\mathbb{I}\mathbb{B}_n \setminus (\Omega \times \mathbb{I}\mathbb{B}_m)$ and $\Omega \times (\mathbb{I}\mathbb{B}_m \setminus \Omega')$ then for all $\omega \in \Omega$ and $\omega_0 \in \Omega \times \Omega'$*

$$\begin{aligned} & K_i(\mathcal{C}_0((\Omega \times \Omega') \setminus \{\omega_0\}, F)) \approx \\ & \approx K_i(\mathcal{C}_0(\Omega \setminus \{\omega\}, F)) \times K_{i+1}(\mathcal{C}_0(\Omega \times (\mathbb{I}\mathbb{B}_m \setminus \Omega'), F)) . \end{aligned}$$

In particular if there is a $p \in \mathbb{N}$ such that $\mathbb{I}\mathbb{B}_m \setminus \Omega'$ is homeomorphic to p copies of \mathbb{R}^m then

$$\begin{aligned} & K_i(\mathcal{C}_0((\Omega \times \Omega') \setminus \{\omega_0\}, F)) \approx \\ & \approx K_i(\mathcal{C}_0(\Omega \setminus \{\omega\}, F)) \times K_{i+m+1}(\mathcal{C}(\Omega, F))^p . \end{aligned}$$

By Theorem 3.1.2 b) and the Product Theorem (Proposition 2.3.1 a)),

$$\begin{aligned} & K_i(\mathcal{C}_0((\Omega \times \Omega') \setminus \{\omega_0\}, F)) \approx K_{i+1}(\mathcal{C}_0(\mathbb{I}\mathbb{B}_n \setminus (\Omega \times \Omega'), F)) \approx \\ & \approx K_{i+1}(\mathcal{C}_0(\mathbb{I}\mathbb{B}_n \setminus (\Omega \times \mathbb{I}\mathbb{B}_m), F)) \times K_{i+1}(\mathcal{C}_0(\Omega \times (\mathbb{I}\mathbb{B}_m \setminus \Omega'), F)) . \end{aligned}$$

By Theorem 3.1.2 b) and Corollary 3.1.5,

$$\begin{aligned} & K_{i+1}(\mathcal{C}_0(\mathbb{I}\mathbb{B}_n \setminus (\Omega \times \mathbb{I}\mathbb{B}_m), F)) \approx K_i(\mathcal{C}_0((\Omega \times \mathbb{I}\mathbb{B}_m) \setminus \{\omega_0\}, F)) \approx \\ & \approx K_i(\mathcal{C}_0(\Omega \setminus \{\omega\}, F)) \end{aligned}$$

and so

$$\begin{aligned} & K_i(\mathcal{C}_0((\Omega \times \Omega') \setminus \{\omega_0\}, F)) \approx \\ & \approx K_i(\mathcal{C}_0(\Omega \setminus \{\omega\}, F)) \times K_{i+1}(\mathcal{C}_0(\Omega \times (\mathbb{I}\mathbb{B}_m \setminus \Omega'), F)) . \end{aligned}$$

We prove now the last assertion. By Theorem 3.1.2 a),

$$K_{i+1}(\mathcal{C}_0(\mathbb{R}^m, \mathcal{C}(\Omega, F))) \approx K_{i+m+1}(\mathcal{C}(\Omega, F))$$

so by the Product Theorem (Proposition 2.3.1 a)),

$$\begin{aligned} & K_{i+1}(\mathcal{C}_0(\Omega \times (\mathbb{I}\mathbb{B}_m \setminus \Omega'), F)) \approx K_{i+1}(\mathcal{C}_0(\mathbb{I}\mathbb{B}_m \setminus \Omega', \mathcal{C}(\Omega, F))) \approx \\ & \approx K_{i+1}(\mathcal{C}_0(\mathbb{R}^m, \mathcal{C}(\Omega, F)))^p \approx K_{i+m+1}(\mathcal{C}(\Omega, F))^p , \\ & K_i(\mathcal{C}_0((\Omega \times \Omega') \setminus \{\omega_0\}, F)) \approx \\ & \approx K_i(\mathcal{C}_0(\Omega \setminus \{\omega\}, F)) \times K_{i+m+1}(\mathcal{C}(\Omega, F))^p . \end{aligned}$$

■

COROLLARY 3.2.12 *Let Ω be a connected graph contained in \mathbb{B}_2 and containing \mathbf{S}_1 , r_0 and r_1 the number of vertices and chords of Ω , respectively, and Γ a nonempty finite subset of $\mathbf{S}_n \times \Omega$. Then*

$$\begin{aligned} K_i(\mathcal{C}_0((\mathbf{S}_n \times \Omega) \setminus \Gamma, F)) &\approx \\ &\approx K_{i+n}(F) \times K_{i+1+n}(F)^{1-r_0+r_1} \times K_{i+1}(F)^{r_1-r_0+Card\Gamma} . \end{aligned}$$

Assume first $\Gamma = \{\omega_0\}$ for some $\omega_0 \in \mathbf{S}_n \times \Omega$. There is an embedding of $\mathbf{S}_n \times \Omega$ in \mathbb{B}_{n+2} such that $\mathbb{B}_{n+2} \setminus (\mathbf{S}_n \times \Omega)$ is homeomorphic to the topological sum of $\mathbb{B}_{n+2} \setminus (\mathbf{S}_n \times \mathbb{B}_2)$ and $\mathbf{S}_n \times (\mathbb{B}_2 \setminus \Omega)$. Since $\mathbf{S}_n \times (\mathbb{B}_2 \setminus \Omega)$ is homeomorphic to $1 - r_0 + r_1$ copies of $\mathbf{S}_n \times \mathbb{R}^2$, we get by Proposition 3.2.11, for $\omega \in \mathbf{S}_n$,

$$\begin{aligned} K_i(\mathcal{C}_0((\mathbf{S}_n \times \Omega) \setminus \{\omega_0\}, F)) &\approx \\ &\approx K_i(\mathcal{C}_0(\mathbf{S}_n \setminus \{\omega\}, F)) \times K_{i+1}(\mathcal{C}(\mathbf{S}_n, F))^{1-r_0+r_1} . \end{aligned}$$

By Theorem 3.2.2 a),b),

$$K_i(\mathcal{C}_0((\mathbf{S}_n \times \Omega) \setminus \{\omega_0\}, F)) \approx K_{i+n}(F) \times K_{i+1+n}(F)^{1-r_0+r_1} \times K_{i+1}(F)^{1-r_0+r_1} .$$

By Proposition 2.4.11,

$$\begin{aligned} K_i(\mathcal{C}_0((\mathbf{S}_n \times \Omega) \setminus \Gamma, F)) &\approx \\ &\approx K_i(\mathcal{C}_0((\mathbf{S}_n \times \Omega) \setminus \{\omega_0\}, F)) \times K_{i+1}(F)^{Card\Gamma-1} \approx \\ &\approx K_{i+n}(F) \times K_{i+1+n}(F)^{1-r_0+r_1} \times K_{i+1}(F)^{r_1-r_0+Card\Gamma} . \end{aligned}$$

■

COROLLARY 3.2.13 *If*

$$\Omega := \mathbf{S}_{n-1} \cup \left(\bigcup_{j \in \mathbb{N}_n} \{ \alpha \in \mathbb{B}_n \mid \alpha_j = 0 \} \right) ,$$

$m \in \mathbb{N}$, and Γ is a finite subset of $\mathbf{S}_m \times \Omega$ then

$$\begin{aligned} K_i(\mathcal{C}_0((\mathbf{S}_m \times \Omega) \setminus \Gamma, F)) &\approx \\ &\approx K_{i+m}(F) \times K_{i+n+1}(F)^{2^n} \times K_{i+m+n+1}(F)^{2^n} \times K_{i+1}(F)^{Card\Gamma-1} . \end{aligned}$$

Assume first $\Gamma = \{\omega_0\}$ for some $\omega_0 \in \mathbf{S}_m \times \Omega$. There is an embedding of $\mathbf{S}_m \times \Omega$ in \mathbf{IB}_{m+n+1} such that $\mathbf{IB}_{m+n+1} \setminus (\mathbf{S}_m \times \Omega)$ is homeomorphic to the topological sum of $\mathbf{IB}_{m+n+1} \setminus (\mathbf{S}_m \times \mathbf{IB}_n)$ and $\mathbf{S}_m \times (\mathbf{IB}_n \setminus \Omega)$. Since $\mathbf{IB}_n \setminus \Omega$ is homeomorphic to the topological sum of 2^n copies of \mathbb{R}^n , by Proposition 3.2.11, for $\omega \in \mathbf{S}_m$,

$$\begin{aligned} K_i(\mathcal{C}_0((\mathbf{S}_m \times \Omega) \setminus \{\omega_0\}, F)) &\approx \\ &\approx K_i(\mathcal{C}_0(\mathbf{S}_m \setminus \{\omega\}, F)) \times K_{i+n+1}(\mathcal{C}(\mathbf{S}_m, F))^{2^n}. \end{aligned}$$

By Proposition 3.2.2 a),b),

$$K_i(\mathcal{C}_0((\mathbf{S}_n \times \Omega) \setminus \{\omega_0\}, F)) \approx K_{i+m}(F) \times K_{i+1+n}(F)^{2^n} \times K_{i+1+m+n}(F)^{2^n}.$$

By Proposition 2.4.11,

$$\begin{aligned} K_i(\mathcal{C}_0((\mathbf{S}_m \times \Omega) \setminus \Gamma, F)) &\approx \\ &\approx K_i(\mathcal{C}_0((\mathbf{S}_m \times \Omega) \setminus \{\omega_0\}, F)) \times K_{i+1}(F)^{\text{Card}\Gamma-1} \approx \\ &\approx K_{i+m}(F) \times K_{i+n+1}(F)^{2^n} \times K_{i+m+n+1}(F)^{2^n} \times K_{i+1}(F)^{\text{Card}\Gamma-1}. \end{aligned}$$

■

LEMMA 3.2.14 *Let $(k_j)_{j \in \mathbb{N}_n}$ be a family in \mathbb{N} , $n \neq 1$, and $m := 1 + \sum_{j \in \mathbb{N}_n} k_j$. There is an embedding of $\prod_{j \in \mathbb{N}_n} \mathbf{S}_{k_j}$ in \mathbf{IB}_m such that $\mathbf{IB}_m \setminus \prod_{j \in \mathbb{N}_n} \mathbf{S}_{k_j}$ has two connected components: one is homeomorphic to $\mathbb{R}^{1+k_n} \times \prod_{j \in \mathbb{N}_{n-1}} \mathbf{S}_{k_j}$ and the other is homeomorphic to $\mathbf{IB}_m \setminus \left(\mathbf{IB}_{1+k_n} \times \prod_{j \in \mathbb{N}_{n-1}} \mathbf{S}_{k_j} \right)$.*

We prove the assertion by induction with respect to $n \in \mathbb{N} \setminus \{1\}$. Assume first $n = 2$, put

$$\Gamma := \left\{ \alpha \in \mathbf{IB}_m \mid \|\alpha\| = \frac{1}{2}, \alpha_{2+k_1} = \alpha_{3+k_1} = \dots = \alpha_m = 0 \right\},$$

and for every $\alpha \in \mathbf{IB}_m$ denote by $d(\alpha)$ the distance of α to Γ . Then

$$\left\{ \alpha \in \mathbf{IB}_m \mid d(\alpha) = \frac{1}{4} \right\}$$

is an embedding of $\mathbf{S}_{k_1} \times \mathbf{S}_{k_2}$ in \mathbf{IB}_m with the desired properties.

Let now $n > 2$ and assume the assertion holds for $n - 1$. Let Γ be a closed set of \mathbb{B}_{m-k_n} homeomorphic to $\prod_{j \in \mathbb{N}_{n-1}} \mathbf{S}_{k_j}$. We may assume $\Gamma \subset \mathbf{S}_{m-k_m}$. We denote for every $\alpha \in \mathbb{B}_m$ by $d(\alpha)$ the distance of α to $\frac{1}{2}\Gamma$. Then $\{\alpha \in \mathbb{B}_m \mid d(\alpha) = \frac{1}{4}\}$ is an embedding with the desired properties. \blacksquare

PROPOSITION 3.2.15 *Let $(k_j)_{j \in \mathbb{N}_n}$ be a family in \mathbb{N} .*

$$\begin{aligned}
 a) \quad & \prod_{j=1}^n \mathbf{S}_{k_j} \in \Upsilon, \quad \mathbb{R}_\Upsilon \subset \left(\prod_{j=1}^n \mathbf{S}_{k_j} \right)_\Upsilon, \\
 & K_i \left(\mathcal{C} \left(\prod_{j=1}^n \mathbf{S}_{k_j}, F \right) \right) \approx \\
 & \approx \begin{cases} K_i(F)^{2^n} & \text{if } \text{all } (k_j)_{j \in \mathbb{N}_n} \text{ are even} \\ \left(K_i(F) \times K_{i+1}(F) \right)^{2^{n-1}} & \text{if } \text{not all } (k_j)_{j \in \mathbb{N}_n} \text{ are even} \end{cases}.
 \end{aligned}$$

b) *If Γ is a nonempty finite subset of $\prod_{j \in \mathbb{N}_n} \mathbf{S}_{k_j}$ then*

$$\begin{aligned}
 & K_i \left(\mathcal{C}_0 \left(\prod_{j \in \mathbb{N}_n} \mathbf{S}_{k_j} \setminus \Gamma, F \right) \right) \approx \\
 & \approx \begin{cases} K_i(F)^{2^{n-1}} \times K_{i+1}(F)^{\text{Card } \Gamma - 1} & \text{if } \text{all } k_j \text{ are even} \\ K_i(F)^{2^{n-1}-1} \times K_{i+1}(F)^{2^{n-1} + \text{Card } \Gamma - 2} & \text{if } \text{not all } k_j \text{ are even} \end{cases}.
 \end{aligned}$$

a) By Theorem 3.2.2 b), $\mathbf{S}_{k_j} \in \Upsilon$, $\mathbb{R}_\Upsilon \subset (\mathbf{S}_{k_j})_\Upsilon$ for every $j \in J$ so by Proposition 1.5.11 a),f), $\prod_{j=1}^n \mathbf{S}_{k_j} \in \Upsilon$, $\mathbb{R}_\Upsilon \subset \left(\prod_{j=1}^n \mathbf{S}_{k_j} \right)_\Upsilon$. By Theorem 3.2.2 b), with the notation of Proposition 1.5.11 a),f),

$$p_j = \frac{3 + (-1)^{k_j}}{2}, \quad q_j = \frac{1 - (-1)^{k_j}}{2}, \quad p_j + q_j = 2, \quad p_j - q_j = 1 + (-1)^{k_j},$$

$$p_J = \frac{1}{2} \left(2^n + \prod_{j=1}^n (1 + (-1)^{k_j}) \right), \quad q_J = \frac{1}{2} \left(2^n - \prod_{j=1}^n (1 + (-1)^{k_j}) \right),$$

and this implies the result.

b) Assume first $\Gamma = \{\omega_0\}$ for some $\omega_0 \in \prod_{j \in \mathbb{N}_n} \mathbf{S}_{k_j}$. We prove the assertion by induction with respect to $n \in \mathbb{N}$. For $n = 1$ this follows from Theorem 3.2.2 e_1). Let $n \neq 1$ and assume the assertion holds for $n - 1$. By Lemma 3.2.14, $\mathbb{I}\mathbf{B}_m \setminus \prod_{j \in \mathbb{N}_n} \mathbf{S}_{k_j}$ is homeomorphic to the topological sum of $\mathbb{R}^{1+k_n} \times \prod_{j \in \mathbb{N}_{n-1}} \mathbf{S}_{k_j}$ and $\mathbb{I}\mathbf{B}_m \setminus \left(\mathbb{I}\mathbf{B}_{1+k_n} \times \prod_{j \in \mathbb{N}_{n-1}} \mathbf{S}_{k_j} \right)$. By Proposition 3.2.11, for $\omega \in \prod_{j \in \mathbb{N}_{n-1}} \mathbf{S}_{k_j}$,

$$\begin{aligned} & K_i \left(\mathcal{C}_0 \left(\prod_{j \in \mathbb{N}_n} \mathbf{S}_{k_j} \setminus \{\omega_0\}, F \right) \right) \approx \\ & \approx K_i \left(\mathcal{C}_0 \left(\prod_{j \in \mathbb{N}_{n-1}} \mathbf{S}_{k_j} \setminus \{\omega\}, F \right) \right) \times \\ & \times K_{i+1} \left(\mathcal{C}_0 \left((\mathbb{I}\mathbf{B}_m \setminus \mathbf{S}_{k_n}) \times \prod_{j \in \mathbb{N}_{n-1}} \mathbf{S}_{k_j}, F \right) \right). \end{aligned}$$

By a) and Theorem 3.2.2 g),

$$\begin{aligned} & K_{i+1} \left(\mathcal{C}_0 \left((\mathbb{I}\mathbf{B}_n \setminus \mathbf{S}_{k_n}) \times \prod_{j \in \mathbb{N}_{n-1}} \mathbf{S}_{k_j}, F \right) \right) \approx \\ & \approx K_{i+k_n} \left(\mathcal{C} \left(\prod_{j \in \mathbb{N}_{n-1}} \mathbf{S}_{k_j}, F \right) \right) \approx \\ & \approx \begin{cases} K_{i+k_n}(F)^{2^{n-1}} & \text{if all } (k_j)_{j \in \mathbb{N}_{n-1}} \text{ are even} \\ (K_i(F) \times K_{i+1}(F))^{2^{n-2}} & \text{if not all } (k_j)_{j \in \mathbb{N}_{n-1}} \text{ are even} \end{cases}. \end{aligned}$$

By the induction hypothesis,

$$\begin{aligned} & K_i \left(\mathcal{C}_0 \left(\prod_{j \in \mathbb{N}_{n-1}} \mathbf{S}_{k_j} \setminus \{\omega\}, F \right) \right) \approx \\ & \approx \begin{cases} K_i(F)^{2^{n-1}-1} & \text{if all } (k_j)_{j \in \mathbb{N}_{n-1}} \text{ are even} \\ K_i(F)^{2^{n-2}-1} \times K_{i+1}(F)^{2^{n-2}} & \text{if not all } (k_j)_{j \in \mathbb{N}_{n-1}} \text{ are even} \end{cases} \end{aligned}$$

so

$$K_i \left(\mathcal{C}_0 \left(\prod_{j \in \mathbb{N}_n} \mathbf{S}_{k_j} \setminus \{\omega_0\}, F \right) \right) \approx$$

$$\approx \begin{cases} K_i(F)^{2^n-1} & \text{if all } (k_j)_{j \in \mathbb{N}_n} \text{ are even} \\ K_i(F)^{2^{n-1}-1} \times K_{i+1}(F)^{2^{n-1}} & \text{if not all } (k_j)_{j \in \mathbb{N}_n} \text{ are even} \end{cases}.$$

This finishes the inductive proof.

We prove now the general case and put $\Omega := \prod_{j \in \mathbb{N}_n} \mathbf{S}_{k_j}$. Since it is possible to find a closed set Δ of Ω such that $\Gamma \subset \Delta$ and $\Delta \setminus \{\omega_0\}$ is K -null, the assertion follows from Proposition 2.4.11. ■

3.3 Some Morphisms

PROPOSITION 3.3.1 *We put*

$$\begin{aligned} \vartheta : \mathbb{B}_n &\longrightarrow \mathbb{B}_n, & (\alpha_j)_{j \in \mathbb{N}_n} &\longmapsto (\alpha_1, \dots, \alpha_{n-1}, -\alpha_n), \\ \vartheta' : \mathbb{R}^n &\longrightarrow \mathbb{R}^n, & (\alpha_j)_{j \in \mathbb{N}_n} &\longmapsto (\alpha_1, \dots, \alpha_{n-1}, -\alpha_n), \\ \vartheta'' : \mathbf{S}_{n-1} &\longrightarrow \mathbf{S}_{n-1}, & (\alpha_j)_{j \in \mathbb{N}_n} &\longmapsto (\alpha_1, \dots, \alpha_{n-1}, -\alpha_n), \\ \phi : \mathcal{C}(\mathbb{B}_n, F) &\longrightarrow \mathcal{C}(\mathbb{B}_n, F), & x &\longmapsto x \circ \vartheta, \\ \phi' : \mathcal{C}_0(\mathbb{R}^n, F) &\longrightarrow \mathcal{C}_0(\mathbb{R}^n, F), & x &\longmapsto x \circ \vartheta', \\ \phi'' : \mathcal{C}(\mathbf{S}_{n-1}, F) &\longrightarrow \mathcal{C}(\mathbf{S}_{n-1}, F), & x &\longmapsto x \circ \vartheta''. \end{aligned}$$

- a) $K_i(\phi) : K_i(\mathcal{C}(\mathbb{B}_n, F)) \longrightarrow K_i(\mathcal{C}(\mathbb{B}_n, F)), \quad a \longmapsto a.$
- b) $K_i(\phi') : K_i(\mathcal{C}_0(\mathbb{R}^n, F)) \longrightarrow K_i(\mathcal{C}_0(\mathbb{R}^n, F)), \quad b \longmapsto -b.$
- c)

$$\begin{aligned} K_i(\phi'') : K_i(\mathcal{C}(\mathbf{S}_{n-1}, F)) &\longrightarrow K_i(\mathcal{C}(\mathbf{S}_{n-1}, F)), \\ (a, b) &\longmapsto \begin{cases} (b, a) & \text{if } n = 1 \\ (a, -b) & \text{if } n > 1 \end{cases}, \end{aligned}$$

where we identified $K_i(\mathcal{C}(\mathbf{S}_{n-1}, F))$ with

$$K_i(\mathcal{C}(\mathbb{B}_n, F)) \times K_{i+1}(\mathcal{C}_0(\mathbb{B}_n \setminus \mathbf{S}_{n-1}, F))$$

using the group isomorphism of Corollary 3.2.8 d) if $n > 1$.

a) follows from the homotopy axiom (Axiom 1.2.5) since ϕ is homotopic to the identity map of $\mathcal{C}(\mathbb{B}_n, F)$.

b) We identify \mathbb{R}^n with the homeomorphic space $\mathbb{B}_n \setminus \mathbf{S}_{n-1}$.

Assume first $n = 1$. Put

$$\psi : \mathcal{C}(\mathbb{B}_1, F) \longrightarrow \mathcal{C}(\{-1, 1\}, F), \quad x \longmapsto x|_{\{-1, 1\}}$$

and denote by $\varphi : \mathcal{C}_0([-1, 1[, F) \longrightarrow \mathcal{C}(\mathbb{B}_1, F)$ the inclusion map and by δ_i the index maps associated to the exact sequence in \mathfrak{M}_E

$$0 \longrightarrow \mathcal{C}_0([-1, 1[, F) \xrightarrow{\varphi} \mathcal{C}(\mathbb{B}_1, F) \xrightarrow{\psi} \mathcal{C}(\{-1, 1\}, F) \longrightarrow 0.$$

By Corollary 2.4.7, $K_i(\psi)a = (a, a)$ for every $a \in K_i(\mathcal{C}(\mathbb{B}_1, F))$ so by the six-term axiom (Axiom 1.2.7),

$$\delta_i(a + b, a + b) = 0, \quad \delta_i(a, b) = -\delta_i(b, a)$$

for all $(a, b) \in K_i(\mathcal{C}(\{-1, 1\}, F))$. By the commutativity of the index maps (Axiom 1.2.8), $K_{i+1}(\phi') \circ \delta_i = \delta_i \circ K_i(\phi'')$. For $(a, b) \in K_i(\mathcal{C}(\{-1, 1\}, F))$, by the above,

$$K_{i+1}(\phi')\delta_i(a, b) = \delta_i K_i(\phi'')(a, b) = \delta_i(b, a) = -\delta_i(a, b).$$

Since δ_i is surjective (because φ factorizes through null and is therefore K-null), $K_i(\phi')b = -b$ for all $b \in K_i(\mathcal{C}_0([-1, 1[, F))$.

If $n > 1$ then the assertion follows from the case $n = 1$, since $\mathcal{C}_0(\mathbb{R}^n, F) \approx \mathcal{C}_0(\mathbb{R}, \mathcal{C}_0(\mathbb{R}^{n-1}, F))$

c) follows from a), b), and Corollary 3.2.8 c). ■

COROLLARY 3.3.2 *If we put*

$$\vartheta : \mathbb{B}_n \longrightarrow \mathbb{B}_n, \quad \alpha \longmapsto -\alpha,$$

$$\vartheta' : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad \alpha \longmapsto -\alpha,$$

$$\vartheta'' : \mathbf{S}_{n-1} \longrightarrow \mathbf{S}_{n-1}, \quad \alpha \longmapsto -\alpha,$$

$$\phi : \mathcal{C}(\mathbb{B}_n, F) \longrightarrow \mathcal{C}(\mathbb{B}_n, F), \quad x \longmapsto x \circ \vartheta,$$

$$\begin{aligned}\phi' : \mathcal{C}_0(\mathbb{R}^n, F) &\longrightarrow \mathcal{C}_0(\mathbb{R}^n, F), \quad x \longmapsto x \circ \vartheta', \\ \phi'' : \mathcal{C}(\mathbf{S}_{n-1}, F) &\longrightarrow \mathcal{C}(\mathbf{S}_{n-1}, F), \quad x \longmapsto x \circ \vartheta''\end{aligned}$$

then

$$\begin{aligned}K_i(\phi) : K_i(\mathcal{C}(\mathbb{B}_n, F)) &\longrightarrow K_i(\mathcal{C}(\mathbb{B}_n, F)), \quad a \longmapsto a, \\ K_i(\phi') : K_i(\mathcal{C}_0(\mathbb{R}^n, F)) &\longrightarrow K_i(\mathcal{C}_0(\mathbb{R}^n, F)), \quad b \longmapsto (-1)^n b, \\ K_i(\phi'') : K_i(\mathcal{C}(\mathbf{S}_{n-1}, F)) &\longrightarrow K_i(\mathcal{C}(\mathbf{S}_{n-1}, F)), \\ (a, b) &\longmapsto \begin{cases} (b, a) & \text{if } n = 1 \\ (a, (-1)^{n+1} b) & \text{if } n > 1 \end{cases},\end{aligned}$$

where we identified $K_i(\mathcal{C}(\mathbf{S}_{n-1}, F))$ with

$$K_i(\mathcal{C}(\mathbb{B}_n, F)) \times K_{i+1}(\mathcal{C}_0(\mathbb{B}_n \setminus \mathbf{S}_{n-1}, F))$$

using the group isomorphism of Corollary 3.2.8 c) if $n > 1$.

The assertion for $K_i(\phi)$ follows from the homotopy axiom (Axiom 1.2.5) since ϕ is homotopic to the identity map of $\mathcal{C}(\mathbb{B}_n, F)$. If n is even then the same holds for $K_i(\phi')$. Assume now n odd and let us denote by $\bar{\phi}'$ the map denoted by ϕ' in Proposition 3.3.1. Then $\phi' \circ \bar{\phi}'$ is homotopic to the identity map of $\mathcal{C}_0(\mathbb{R}^n, F)$ so by Corollary 3.3.1, for every $b \in K_i(\mathcal{C}_0(\mathbb{R}^n, F))$,

$$K_i(\phi')b = -K_i(\phi')K_i(\bar{\phi}')b = -b = (-1)^n b.$$

The assertion for $K_i(\phi'')$ follows from the corresponding assertions for $K_i(\phi)$ and $K_i(\phi')$ and from Corollary 3.2.8 d). ■

PROPOSITION 3.3.3 *Let $\alpha, \beta \in [0, 2\pi[$, $\alpha < \beta$, $\Omega := \{e^{i\omega} \mid \omega \in]\alpha, \beta[\}$, $\Gamma := \mathbb{T} \setminus \Omega$, $\varphi : \mathcal{C}_0(\Omega, F) \longrightarrow \mathcal{C}(\mathbb{T}, F)$ the inclusion map, and*

$$\begin{aligned}\psi : \mathcal{C}(\mathbb{T}, F) &\longrightarrow \mathcal{C}(\Gamma, F), \quad x \longmapsto x|_{\Gamma}, \\ \bar{\psi} : \mathcal{C}(\Gamma, F) &\longrightarrow F, \quad x \longmapsto x(1), \\ \vartheta :]0, 2\pi[\longrightarrow]\alpha, \beta[, \quad \omega &\longmapsto \frac{\beta - \alpha}{2\pi} \omega + \alpha.\end{aligned}$$

For every $x \in \mathcal{C}_0(\Omega, F)$ put

$$\tilde{x} : \mathbb{T} \longrightarrow F, \quad e^{i\omega} \longmapsto \begin{cases} x(e^{i\vartheta(\omega)}) & \text{if } \omega \in]0, 2\pi[\\ 0 & \text{if } \omega \in \{0, 2\pi\} \end{cases}$$

and define

$$\phi : \mathcal{C}_0(\Omega, F) \longrightarrow \mathcal{C}_0(\mathbb{T} \setminus \{1\}, F), \quad x \longmapsto \tilde{x}.$$

a) $K_i(\phi)$ and $K_i(\tilde{\psi})$ are group isomorphisms and so

$$K_i(\mathcal{C}_0(\Omega, F)) \approx K_{i+1}(F), \quad K_i(\mathcal{C}(\Gamma, F)) \approx K_i(F).$$

b) If we identify $K_i(\mathcal{C}_0(\Omega, F))$ with $K_{i+1}(F)$ and $K_i(\mathcal{C}(\Gamma, F))$ with $K_i(F)$ using the isomorphisms from a) and $K_i(\mathcal{C}(\mathbb{T}, F))$ with $K_i(F) \times K_{i+1}(F)$ using e.g. Alexandroff K -theorem (Theorem 2.2.1 a)) then

$$K_i(\phi) : K_i(\mathcal{C}_0(\Omega, F)) \longrightarrow K_i(\mathcal{C}(\mathbb{T}, F)), \quad b \longmapsto (0, b),$$

$$K_i(\psi) : K_i(\mathcal{C}(\mathbb{T}, F)) \longrightarrow K_i(\mathcal{C}(\Gamma, F)), \quad (a, b) \longmapsto a.$$

a) ϕ is an E - C^* -isomorphism. Put

$$\tilde{\psi} : F \longrightarrow \mathcal{C}(\Gamma, F), \quad x \longmapsto 1_{\mathcal{C}(\Gamma, F)} x.$$

Then $\mathcal{C}(\Gamma, F) \xrightarrow{\tilde{\psi}} F \xrightarrow{\tilde{\psi}} \mathcal{C}(\Gamma, F)$ is a homotopy in \mathfrak{M}_E so $K_i(\phi)$ and $K_i(\tilde{\psi})$ are group isomorphisms by the homotopy axiom (Axiom 1.2.5). The last assertion follows now from Theorem 3.2.2 a).

b) For every $s \in [0, 1]$ put

$$\vartheta_s : \mathbb{T} \longrightarrow \mathbb{T}, \quad e^{i\omega} \longmapsto \begin{cases} e^{is\omega} & \text{if } \omega \in [0, \alpha] \\ e^{is\omega} e^{\frac{2\pi i(1-s)(\omega-\alpha)}{\beta-\alpha}} & \text{if } \omega \in]\alpha, \beta[\\ e^{is\omega} e^{2\pi i(1-s)} & \text{if } \omega \in [\beta, 2\pi] \end{cases},$$

$$\phi_s : \mathcal{C}(\mathbb{T}, F) \longrightarrow \mathcal{C}(\mathbb{T}, F), \quad x \longmapsto x \circ \vartheta_s.$$

Then $(\phi_s)_{s \in [0, 1]}$ is a pointwise continuous path in $\mathcal{C}(\mathbb{T}, F)$ such that ϕ_1 is the identity map. By the homotopy axiom (Axiom 1.2.5), $K_i(\phi_0)$ is the identity map of $K_i(\mathcal{C}(\mathbb{T}, F))$. Let

$$\phi' : \mathcal{C}_0(\mathbb{T} \setminus \{1\}, F) \longrightarrow \mathcal{C}(\mathbb{T}, F)$$

be the inclusion map and

$$\psi' : \mathcal{C}(\mathbb{I}, F) \longrightarrow F, \quad x \longmapsto x(1).$$

Then $\phi_0 \circ \varphi = \varphi' \circ \phi$ and $\psi' \circ \phi_0 = \bar{\psi} \circ \psi$ so (by a)) for $a \in K_i(F)$ and $b \in K_{i+1}(F)$,

$$K_i(\varphi)b = K_i(\phi_0)K_i(\varphi)b = K_i(\varphi')K_i(\phi)b = K_i(\varphi')b = (0, b),$$

$$K_i(\psi)(a, b) = K_i(\bar{\psi})K_i(\psi)(a, b) = K_i(\psi')K_i(\phi_0)(a, b) = K_i(\psi')(a, b) = a$$

by the Alexandroff K-theorem (Theorem 2.2.1 a)). ■

PROPOSITION 3.3.4 Put $\Gamma := \left\{ e^{\frac{2\pi i j}{n}} \mid j \in \mathbb{N}_n \right\}$ and

$$\psi : \mathcal{C}(\mathbb{I}, F) \longrightarrow \mathcal{C}(\Gamma, F), \quad x \longmapsto x|_{\Gamma},$$

and denote by

$$\varphi : \mathcal{C}_0(\mathbb{I} \setminus \Gamma, F) \longrightarrow \mathcal{C}(\mathbb{I}, F)$$

the inclusion map and by δ_i the index maps associated to the exact sequence in \mathfrak{M}_E

$$0 \longrightarrow \mathcal{C}_0(\mathbb{I} \setminus \Gamma, F) \xrightarrow{\varphi} \mathcal{C}(\mathbb{I}, F) \xrightarrow{\psi} \mathcal{C}(\Gamma, F) \longrightarrow 0.$$

$$a) \ K_i(\mathcal{C}_0(\mathbb{I} \setminus \Gamma, F)) \approx K_{i+1}(F)^n, \quad K_i(\mathcal{C}(\Gamma, F)) \approx K_i(F)^n.$$

b) We identify the isomorphic groups of a) and identify $K_i(\mathcal{C}(\mathbb{I}, F))$ with $K_i(F) \times K_{i+1}(F)$ (Theorem 3.2.2 b)).

$$K_i(\varphi) : K_i(\mathcal{C}_0(\mathbb{I} \setminus \Gamma, F)) \longrightarrow K_i(\mathcal{C}(\mathbb{I}, F)), \quad (b_j)_{j \in \mathbb{N}_n} \longmapsto \left(0, \sum_{j \in \mathbb{N}_n} b_j \right),$$

$$K_i(\psi) : K_i(\mathcal{C}(\mathbb{I}, F)) \longrightarrow K_i(\mathcal{C}(\Gamma, F)), \quad (a, b) \longmapsto (a)_{j \in \mathbb{N}_n}.$$

If $n = 2$ and $K_i(F)$ is isomorphic to \mathbb{Z} or to \mathbb{Z}_p for some $p \in \mathbb{N}$ or to the group of rational numbers then there is an automorphism

$$\Phi_i : K_i(F) \longrightarrow K_i(F)$$

such that

$$\delta_i : K_i(\mathcal{C}(\Gamma, F)) \longrightarrow K_{i+1}(\mathcal{C}_0(\mathbb{I} \setminus \Gamma, F)), \quad (a, b) \longmapsto (\Phi_i(a - b), \Phi(b - a)).$$

c) If we put

$$\begin{aligned}\vartheta : \mathbb{T} \setminus \Gamma &\longrightarrow \mathbb{T} \setminus \{1\}, & z &\longmapsto z^n, \\ \vartheta' : \mathbb{T} &\longrightarrow \mathbb{T}, & z &\longmapsto z^n, \\ \vartheta'' : \Gamma &\longrightarrow \{1\}, & z &\longmapsto z^n, \\ \phi : \mathcal{C}_0(\mathbb{T} \setminus \{1\}, F) &\longrightarrow \mathcal{C}_0(\mathbb{T} \setminus \Gamma, F), & x &\longmapsto x \circ \vartheta, \\ \phi' : \mathcal{C}(\mathbb{T}, F) &\longrightarrow \mathcal{C}(\mathbb{T}, F), & x &\longmapsto x \circ \vartheta', \\ \phi'' : \mathcal{C}(\{1\}, F) &\longrightarrow \mathcal{C}(\Gamma, F), & x &\longmapsto x \circ \vartheta''\end{aligned}$$

then, with the identifications of a) and b),

$$\begin{aligned}K_i(\phi) : K_i(\mathcal{C}_0(\mathbb{T} \setminus \{1\}, F)) &\longrightarrow K_i(\mathcal{C}_0(\mathbb{T} \setminus \Gamma, F)), & b &\longmapsto (b)_{j \in \mathbb{N}_n}, \\ K_i(\phi') : K_i(\mathcal{C}(\mathbb{T}, F)) &\longrightarrow K_i(\mathcal{C}(\mathbb{T}, F)), & (a, b) &\longmapsto (a, nb), \\ K_i(\phi'') : K_i(\mathcal{C}(\{1\}, F)) &\longrightarrow K_i(\mathcal{C}(\Gamma, F)), & a &\longmapsto (a)_{j \in \mathbb{N}_n}.\end{aligned}$$

a) Put $\Omega_j := \left\{ e^{\frac{2\pi i \omega}{n}} \mid \omega \in]j-1, j[\right\}$ for every $j \in \mathbb{N}_n$. By Proposition 3.3.3 a), for every $j \in \mathbb{N}_n$,

$$K_i(\mathcal{C}_0(\Omega_j, F)) \approx K_{i+1}(F).$$

so

$$K_i(\mathcal{C}_0(\mathbb{T} \setminus \Gamma, F)) \approx K_{i+1}(F)^n, \quad K_i(\mathcal{C}(\Gamma, F)) \approx K_i(F)^n$$

by the Product Theorem (Proposition 2.3.1 a)).

b) By Corollary 2.4.7,

$$K_i(\psi) : K_i(\mathcal{C}(\mathbb{T}, F)) \longrightarrow K_i(\mathcal{C}(\Gamma, F)), \quad (a, b) \longmapsto (a)_{j \in \mathbb{N}_n}.$$

If we denote by

$$\varphi_j : \mathcal{C}_0(\Omega_j, F) \longrightarrow \mathcal{C}(\mathbb{T}, F)$$

the inclusion map then

$$K_i(\varphi_j) : K_i(\mathcal{C}_0(\Omega_j, F)) \longrightarrow K_i(\mathcal{C}(\mathbb{T}, F)), \quad b \longmapsto (0, b)$$

by Proposition 3.3.3 b). By Proposition 3.3.3 a) and Corollary 2.3.2,

$$K_i(\phi) : K_i(\mathcal{C}_0(\mathbb{T} \setminus \Gamma, F)) \longrightarrow K_i(\mathcal{C}(\mathbb{T}, F)), \quad (b_j)_{j \in \mathbb{N}_n} \longmapsto \left(0, \sum_{j \in \mathbb{N}_n} b_j \right).$$

In order to prove the last assertion we define $a', b', a'', b'' \in K_i(F)$ by

$$(a', b') := \delta_i(1, 0), \quad (a'', b'') := \delta_i(0, 1).$$

From

$$0 = \delta_i(1, 1) = (a', b') + (a'', b'') = (a' + a'', b' + b'')$$

we get $a'' = -a'$ and $b'' = -b'$. There are $j, k \in \mathbf{Z}$ such that $\delta_i(j, k) = (1, -1)$. Then

$$(1, -1) = \delta_i(j, k) = (ja', kb') - (ka', kb') = ((j-k)a', (j-k)b'),$$

$$(j-k)a' = 1, \quad (j-k)b' = -1.$$

Thus a' is invertible in the ring $K_i(F)$ and $a'^{-1} = j-k$. It follows $b' = -a'$. If we put

$$\Phi_i : K_i(F) \longrightarrow K_i(F), \quad c \longmapsto a'c$$

then Φ_i is an automorphism and for all $a, b \in K_i(F)$,

$$\delta_i(a, b) = (a'a, -a'a) - (a'b, -a'b) = (a'(a-b), a'(b-a)) = (\Phi_i(a-b), \Phi_i(b-a)).$$

c) The assertions for $K_i(\phi)$ and $K_i(\phi'')$ follow from the Product Theorem (Proposition 2.3.1 a)). If $\phi' : \mathcal{C}_0(\mathbb{I} \setminus \{1\}, F) \longrightarrow \mathcal{C}(\mathbb{I}, F)$ denotes the inclusion map and

$$\psi' : \mathcal{C}(\mathbb{I}, F) \longrightarrow \mathcal{C}(\{1\}, F), \quad x \longmapsto x|_{\{1\}}$$

then the diagram

$$\begin{array}{ccccc} K_i(\mathcal{C}_0(\mathbb{I} \setminus \{1\}, F)) & \xrightarrow{K_i(\phi')} & K_i(\mathcal{C}(\mathbb{I}, F)) & \xrightarrow{K_i(\psi')} & K_i(\mathcal{C}(\{1\}, F)) \\ K_i(\phi) \downarrow & & \downarrow K_i(\phi') & & \downarrow K_i(\phi'') \\ K_i(\mathcal{C}_0(\mathbb{I} \setminus \Gamma, F)) & \xrightarrow{K_i(\phi)} & K_i(\mathcal{C}(\mathbb{I}, F)) & \xrightarrow{K_i(\psi)} & K_i(\mathcal{C}(\Gamma, F)) \end{array}$$

is commutative. Let $(a, b) \in K_i(\mathcal{C}(\mathbb{I}, F))$ and put $(a', b') := K_i(\phi')(a, 0)$. By b),

$$(a)_{j \in \mathbb{N}_n} = K_i(\phi'')a = K_i(\phi'')K_i(\psi')(a, 0) =$$

$$= K_i(\psi)K_i(\phi')(a, 0) = K_i(\psi)(a', b') = (a')_{j \in \mathbb{N}_n},$$

$$K_i(\phi')(0, b) = K_i(\phi')K_i(\phi')b = K_i(\phi)K_i(\phi)b = K_i(\phi)(b)_{j \in \mathbb{N}_n} = (0, nb)$$

so $K_i(\phi')(a, b) = (a, nb)$. ■

COROLLARY 3.3.5 *If we put*

$$\begin{aligned}\vartheta : \mathbb{B}_2 &\longrightarrow \mathbb{B}_2, & z &\longmapsto z^n, \\ \vartheta' : \mathbf{C} &\longrightarrow \mathbf{C}, & z &\longmapsto z^n, \\ \vartheta'' : \mathbf{S}_1 &\longrightarrow \mathbf{S}_1, & z &\longmapsto z^n, \\ \phi : \mathcal{C}(\mathbb{B}_2, F) &\longrightarrow \mathcal{C}(\mathbb{B}_2, F), & x &\longmapsto x \circ \vartheta, \\ \phi' : \mathcal{C}_0(\mathbf{C}, F) &\longrightarrow \mathcal{C}_0(\mathbf{C}, F), & x &\longmapsto x \circ \vartheta', \\ \phi'' : \mathcal{C}(\mathbf{S}_1, F) &\longrightarrow \mathcal{C}_0(\mathbf{S}_1, F), & x &\longmapsto x \circ \vartheta''.\end{aligned}$$

then $K_i(\phi)$ is the identity map of $K_i(\mathcal{C}(\mathbb{B}_2, F))$ and

$$\begin{aligned}K_i(\phi') : K_i(\mathcal{C}_0(\mathbf{C}, F)) &\longrightarrow K_i(\mathcal{C}_0(\mathbf{C}, F)), & a &\longmapsto na, \\ K_i(\phi'') : K_i(\mathcal{C}(\mathbf{S}_1, F)) &\longrightarrow K_i(\mathcal{C}(\mathbf{S}_1, F)), & (a, b) &\longmapsto (a, nb),\end{aligned}$$

We identify the homeomorphic spaces \mathbf{C} and $\mathbb{B}_2 \setminus \mathbf{S}_1$. By Corollary 3.2.8 c),

$$K_i(\mathcal{C}(\mathbf{S}_1, F)) \approx K_i(\mathcal{C}(\mathbb{B}_2, F)) \times K_{i+1}(\mathcal{C}_0(\mathbb{B}_2 \setminus \mathbf{S}_1, F))$$

and by Proposition 3.3.4 e),

$$K_i(\phi'') : K_i(\mathcal{C}(\mathbf{S}_1, F)) \longrightarrow K_i(\mathcal{C}(\mathbf{S}_1, F)), \quad (a, b) \longmapsto (a, nb).$$

By Corollary 2.2.2 b) and Theorem 3.1.2 a), $K_i(\phi)$ is the identity map of $K_i(\mathcal{C}(\mathbb{B}_2, F))$ and

$$K_i(\phi') : K_i(\mathcal{C}_0(\mathbb{B}_2 \setminus \mathbf{S}_1, F)) \longrightarrow K_i(\mathcal{C}_0(\mathbb{B}_2 \setminus \mathbf{S}_1, F)), \quad a \longmapsto na. \quad \blacksquare$$

PROPOSITION 3.3.6 *Let $m, n \in \mathbb{N}$ and*

$$\begin{aligned}\vartheta_1 : \mathbb{T} &\longrightarrow \mathbb{T}, & w &\longmapsto w^m, \\ \vartheta_2 : \mathbb{T} &\longrightarrow \mathbb{T}, & z &\longmapsto z^n, \\ \psi : \mathcal{C}(\mathbb{T} \times \mathbb{T}, F) &\longrightarrow \mathcal{C}(\mathbb{T} \times \mathbb{T}, F), & x &\longmapsto x \circ (\vartheta_1 \times \vartheta_2).\end{aligned}$$

We identify $K_i(\mathcal{C}(\mathbb{T}, F'))$ with $K_i(F') \times K_{i+1}(F')$ for all E - C^* -algebras F' by using the group isomorphism of Theorem 3.2.2 b). Let

$$a \in K_i(\mathcal{C}(\mathbb{T} \times \mathbb{T}, F)) \approx K_i(\mathcal{C}(\mathbb{T}, \mathcal{C}(\mathbb{T}, F))) \approx K_i(\mathcal{C}(\mathbb{T}, F)) \times K_{i+1}(\mathcal{C}(\mathbb{T}, F))$$

and put $a_0 \in K_i(\mathcal{C}(\mathbb{I}, F))$, $a_1 \in K_{i+1}(\mathcal{C}(\mathbb{I}, F))$ such that $a = (a_0, a_1)$ and $a_{0,0}, a_{1,1} \in K_i(F)$ and $a_{0,1}, a_{1,0} \in K_{i+1}(F)$ such that $a_0 = (a_{0,0}, a_{0,1})$ and $a_1 = (a_{1,0}, a_{1,1})$. Then

$$K_i(\psi) = ((a_{0,0}, mna_{1,1}), (na_{0,1}, ma_{1,0})) .$$

We put

$$\begin{aligned} \phi : \mathcal{C}(\mathbb{I}, F) &\longrightarrow \mathcal{C}(\mathbb{I}, F) , \quad x \longmapsto x \circ \vartheta_2 , \\ \phi_1 : \mathcal{C}(\mathbb{I}, \mathcal{C}(\mathbb{I}, F)) &\longrightarrow \mathcal{C}(\mathbb{I}, \mathcal{C}(\mathbb{I}, F)) , \quad x \longmapsto x \circ \vartheta_1 , \\ \phi_2 : \mathcal{C}(\mathbb{I}, \mathcal{C}(\mathbb{I}, F)) &\longrightarrow \mathcal{C}(\mathbb{I}, \mathcal{C}(\mathbb{I}, F)) , \quad x \longmapsto \phi \circ x , \end{aligned}$$

By Corollary 3.3.5,

$$K_i(\phi_1)a = (a_0, ma_1), \quad K_i(\phi)a_0 = (a_{0,0}, na_{0,1}), \quad K_{i+1}(\phi)a_1 = (a_{1,0}, na_{1,1})$$

so by Corollary 3.2.8 c), d),

$$K_i(\phi_2)K_i(\phi_1)a = (K_i(\phi)a_0, K_{i+1}(\phi)ma_1) = ((a_{0,0}, na_{0,1}), (ma_{1,0}, mna_{1,1})) .$$

Since $\psi = \phi_2 \circ \phi_1$,

$$K_i(\psi) = ((a_{0,0}, mna_{1,1}), (na_{0,1}, ma_{1,0})) .$$

■

3.4 Some Non-orientable Compact Spaces

DEFINITION 3.4.1 We denote by \mathbb{P}_n the n -dimensional **projective space**, which is obtained from \mathbb{B}_n by identifying α with $-\alpha$ for all $\alpha \in \mathbb{B}_n$ with $\|\alpha\| = 1$.

PROPOSITION 3.4.2 Put

$$\Omega := \mathbb{P}_{n+1} \setminus \{ \alpha \in \mathbb{P}_{n+1} \mid \|\alpha\| = 1, \alpha_{n+1} = 0 \} ,$$

$$\Omega' := \{ \alpha \in \Omega \mid \|\alpha\| = 1 \} ,$$

$$\psi : \mathcal{C}_0(\Omega, F) \longrightarrow \mathcal{C}_0(\Omega', F) , \quad x \longmapsto x|_{\Omega'}$$

and denote by

$$\varphi : \mathcal{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F) \longrightarrow \mathcal{C}_0(\Omega, F)$$

the inclusion map and by δ_i the index maps associated to the exact sequence in \mathfrak{M}_E

$$0 \longrightarrow \mathcal{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F) \xrightarrow{\varphi} \mathcal{C}_0(\Omega, F) \xrightarrow{\psi} \mathcal{C}_0(\Omega', F) \longrightarrow 0 .$$

a) $K_i(\mathcal{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F)) \approx K_{i+n+1}(F)$, $K_i(\mathcal{C}_0(\Omega', F)) \approx K_{i+n}(F)$, and there is an automorphism $\Phi_i : K_{i+n}(F) \longrightarrow K_{i+n}(F)$ such that

$$\begin{aligned} \delta_i : K_i(\mathcal{C}_0(\Omega', F)) &\longrightarrow K_{i+1}(\mathcal{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F)), \\ a &\longmapsto \Phi_i(a - (-1)^n a). \end{aligned}$$

b) If n is even then $\delta_i = 0$, $K_i(\varphi)$ is injective, $K_i(\psi)$ is surjective, and

$$\frac{K_i(\mathcal{C}_0(\Omega, F))}{K_{i+1}(F)} \approx K_i(F).$$

c) If n is odd and for a fixed $i \in \{0, 1\}$

$$a \in K_i(F), 2a = 0 \implies a = 0$$

$$\text{then } K_i(\psi) = 0, K_i(\mathcal{C}_0(\Omega, F)) \approx \frac{K_i(F)}{2K_i(F)},$$

$$K_i(\varphi) : K_i(\mathcal{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F)) \longrightarrow K_i(\mathcal{C}_0(\Omega, F))$$

is the quotient map, and

$$\delta_i : K_i(\mathcal{C}_0(\Omega', F)) \longrightarrow K_{i+1}(\mathcal{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F)), \quad a \longmapsto 2\Phi_i a.$$

a) By Theorem 3.2.2 a), $K_i(\mathcal{C}_0(\mathbb{R}^n, F)) \approx K_{i+n}(F)$. Since $\mathbb{B}_{n+1} \setminus \mathbf{S}_n$ is homeomorphic to \mathbb{R}^{n+1} , $K_i(\mathcal{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F)) \approx K_{i+n+1}(F)$. Since Ω' is homeomorphic to \mathbb{R}^n , $K_i(\mathcal{C}_0(\Omega', F)) \approx K_{i+n}(F)$. We use the notation of Proposition 3.2.9, which we mark by a bar in order to distinguish it from the present notation. Moreover we denote by $\vartheta : \bar{\Omega} \longrightarrow \Omega$ and $\vartheta' : \bar{\Omega}' \longrightarrow \Omega'$ the covering maps and put

$$\begin{aligned} \phi : \mathcal{C}_0(\Omega, F) &\longrightarrow \mathcal{C}_0(\bar{\Omega}, F), \quad x \longmapsto x \circ \vartheta, \\ \phi' : \mathcal{C}_0(\Omega', F) &\longrightarrow \mathcal{C}_0(\bar{\Omega}', F), \quad x \longmapsto x \circ \vartheta'. \end{aligned}$$

By the Product Theorem (Proposition 2.3.1 a)), Proposition 3.2.9 a), and Proposition 3.3.1 b),

$$K_i(\phi') : K_i(\mathcal{C}_0(\Omega', F)) \longrightarrow K_i(\mathcal{C}_0(\bar{\Omega}', F)), \quad a \longmapsto (a, (-1)^n a).$$

By the commutativity of the index maps (Axiom 1.2.8), $\delta_i = \bar{\delta}_i \circ K_i(\phi')$ so by Proposition 3.2.9 b),

$$\delta_i : K_i(\mathcal{C}_0(\Omega', F)) \longrightarrow K_{i+1}(\mathcal{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F)), \quad a \longmapsto \Phi_i(a - (-1)^n a).$$

b) and c) follow from a) and the six-term axiom (Axiom 1.2.7). ■

COROLLARY 3.4.3 *We use the notation and the hypothesis of Proposition 3.4.2, take $n = 1$, put $\Gamma := \{x \in \mathbb{IP}_2 \mid \|x\| = 1\}$,*

$$\psi' : \mathcal{C}(\mathbb{IP}_2, F) \longrightarrow \mathcal{C}(\Gamma, F), \quad x \longmapsto x|_{\Gamma},$$

and denote by $\varphi' : \mathcal{C}_0(\mathbb{IB}_2 \setminus \mathbf{S}_1, F) \longrightarrow \mathcal{C}(\mathbb{IP}_2, F)$ the inclusion map and by δ'_i the index maps associated to the exact sequence in \mathfrak{M}_E

$$0 \longrightarrow \mathcal{C}_0(\mathbb{IB}_2 \setminus \mathbf{S}_1, F) \xrightarrow{\varphi'} \mathcal{C}(\mathbb{IP}_2, F) \xrightarrow{\psi'} \mathcal{C}(\Gamma, F) \longrightarrow 0.$$

Then $K_i(\mathcal{C}(\mathbb{IP}_2, F)) \approx K_i(F) \times \frac{K_i(F)}{2K_i(F)}$, $K_i(\mathcal{C}(\Gamma, F)) \approx K_i(F) \times K_{i+1}(F)$,

$$K_i(\varphi') : K_i(\mathcal{C}_0(\mathbb{IB}_2 \setminus \mathbf{S}_1, F)) \longrightarrow K_i(K_i(\mathcal{C}(\mathbb{IP}_2, F))), \quad a \longmapsto (0, \Phi_i a),$$

$$K_i(\psi') : K_i(\mathcal{C}(\mathbb{IP}_2, F)) \longrightarrow K_i(\mathcal{C}(\Gamma, F)), \quad (a, c) \longmapsto (a, 0),$$

$$\delta'_i : K_i(\mathcal{C}(\Gamma, F)) \longrightarrow K_{i+1}(\mathcal{C}_0(\mathbb{IB}_2 \setminus \mathbf{S}_1, F)), \quad (a, b) \longmapsto 2b. \quad \blacksquare$$

PROPOSITION 3.4.4 *Let*

$$\vartheta : [0, 1] \longrightarrow \mathbb{T}, \quad \omega \longmapsto e^{2\pi i \omega},$$

$$\phi : \mathcal{C}(\mathbb{T}, F) \longrightarrow \mathcal{C}([0, 1], F), \quad x \longmapsto x \circ \vartheta.$$

If we identify $K_i(\mathcal{C}(\mathbb{T}, F))$ with $K_i(F) \times K_{i+1}(F)$ (Theorem 3.2.2 b)) and $K_i(\mathcal{C}([0, 1], F))$ with $K_i(F)$ (Theorem 3.1.2 a)) then

$$K_i(\phi) : K_i(\mathcal{C}(\mathbb{T}, F)) \longrightarrow K_i(\mathcal{C}([0, 1], F)), \quad (a, b) \longmapsto a.$$

Put

$$\vartheta' :]0, 1[\longrightarrow \mathbb{T} \setminus \{1\}, \quad \omega \longmapsto e^{2\pi i \omega},$$

$$\phi' : \mathcal{C}_0(\mathbb{T} \setminus \{1\}, F) \longrightarrow \mathcal{C}_0(]0, 1[, F), \quad x \longmapsto x \circ \vartheta'$$

and denote by

$$\varphi : \mathcal{C}_0(]0, 1[, F) \longrightarrow \mathcal{C}([0, 1], F), \quad \varphi' : \mathcal{C}_0(\mathbb{T} \setminus \{1\}, F) \longrightarrow \mathcal{C}(\mathbb{T}, F)$$

the inclusion maps. Then $\phi \circ \varphi' = \varphi \circ \phi'$, so $K_i(\phi) \circ K_i(\varphi') = K_i(\varphi) \circ K_i(\phi') = 0$, since φ factorizes through 0. Thus $K_i(\phi)(0, b) = 0$ for all $b \in K_{i+1}(F)$.

Put

$$\psi : \mathcal{C}([0, 1], F) \longrightarrow \mathcal{C}(\{0, 1\}, F) \approx F \times F, \quad x \longmapsto x|_{\{0, 1\}},$$

$$\psi' : \mathcal{C}(\mathbb{I}, F) \longrightarrow F, \quad x \longmapsto x(1),$$

$$\mu : F \longrightarrow \mathcal{C}(\{0, 1\}, F), \quad x \longmapsto (x, x).$$

Then $\psi \circ \phi = \mu \circ \psi'$, so $K_i(\psi) \circ K_i(\phi) = K_i(\mu) \circ K_i(\psi')$ and we get (by the above)

$$K_i(\psi)K_i(\phi)(a, b) = K_i(\psi)K_i(\phi)(a, 0) = K_i(\mu)K_i(\psi')(a, 0) = K_i(\mu)a = (a, a),$$

$$K_i(\phi)(a, b) = a$$

for all $(a, b) \in K_i(F) \times K_{i+1}(F)$. ■

DEFINITION 3.4.5 We denote by \mathbf{IM} the **Möbius band** obtained from $[0, 1] \times [-1, 1]$ by identifying the points $(0, \beta)$ and $(1, -\beta)$ for every $\beta \in [-1, 1]$. We put for every $j \in \{-1, 0, 1\}$

$$\Gamma_j^{\mathbf{IM}} := \{ (\alpha, j) \in \mathbf{IM} \mid \alpha \in [0, 1] \}.$$

PROPOSITION 3.4.6 For every $j \in \{-1, 0, 1\}$ put

$$\psi_j : \mathcal{C}(\mathbf{IM}, F) \longrightarrow \mathcal{C}(\Gamma_j^{\mathbf{IM}}, F), \quad x \longmapsto x|_{\Gamma_j^{\mathbf{IM}}}.$$

a) $\Gamma_0^{\mathbf{IM}}$ is homeomorphic to \mathbb{I} and $\Gamma_j^{\mathbf{IM}}$ is homeomorphic to $[0, 1]$ for all $j \in \{-1, 1\}$.

b) $\mathcal{C}_0(\mathbf{IM} \setminus \Gamma_0^{\mathbf{IM}}, F)$ is K -null and

$$K_i(\psi_0) : K_i(\mathcal{C}(\mathbf{IM}, F)) \longrightarrow K_i(\mathcal{C}(\Gamma_0^{\mathbf{IM}}, F)) \approx K_i(F) \times K_{i+1}(F)$$

is a group isomorphism.

c) If we identify $K_i(\mathcal{C}(\mathbf{IM}, F))$ with $K_i(F) \times K_{i+1}(F)$ using the group isomorphism $K_i(\psi_0)$ of b) and $K_i(\mathcal{C}(\Gamma_1^{\mathbf{IM}}, F))$ with $K_i(F)$ using a) (and Theorem 3.1.2 a)) then

$$K_i(\psi_1) : K_i(\mathcal{C}(\mathbf{IM}, F)) \longrightarrow K_i(\mathcal{C}(\Gamma_1^{\mathbf{IM}}, F)), \quad (a, b) \longmapsto a.$$

d) If we put $\omega := (0, 0) = (1, 0) \in \mathbf{IM}$, $\Gamma := \{ (\alpha, 0) \mid \alpha \in]0, 1[\}$, and

$$\psi : \mathcal{C}_0(\mathbf{IM} \setminus \{\omega\}, F) \longrightarrow \mathcal{C}_0(\Gamma, F), \quad x \longmapsto x|_{\Gamma}$$

then

$$K_i(\psi) : K_i(\mathcal{C}_0(\mathbf{IM} \setminus \{\omega\}, F)) \longrightarrow K_i(\mathcal{C}_0(\Gamma, F)) \approx K_{i+1}(F)$$

is a group isomorphism.

e) If Γ' is a finite subset of \mathbb{M} then

$$K_i(\mathcal{C}_0(\mathbb{M} \setminus \Gamma', F)) \approx K_{i+1}(F)^{\Gamma'}$$

a) is easy to see.

b) For every $s \in]0, 1]$ put

$$\vartheta_s : \mathbb{M} \setminus \Gamma_0^{\mathbb{M}} \longrightarrow \mathbb{M} \setminus \Gamma_0^{\mathbb{M}}, \quad (\alpha, \beta) \longmapsto (\alpha, s\beta).$$

By Proposition 2.4.1 (replacing there Ω by $\mathbb{M} \setminus \Gamma_0^{\mathbb{M}}$), $\mathcal{C}_0(\mathbb{M} \setminus \Gamma_0^{\mathbb{M}}, F)$ is K-null and the assertion follows from the Topological six-term sequence (Proposition 2.1.8 a)) and a) (and Theorem 3.2.2 b)).

c) follows from b) and Proposition 3.4.4.

d) If $\varphi : \mathcal{C}_0(\mathbb{M} \setminus \Gamma_0^{\mathbb{M}}, F) \longrightarrow \mathcal{C}_0(\mathbb{M} \setminus \{\omega\}, F)$ denotes the inclusion map then

$$0 \longrightarrow \mathcal{C}_0(\mathbb{M} \setminus \Gamma_0^{\mathbb{M}}, F) \xrightarrow{\varphi} \mathcal{C}_0(\mathbb{M} \setminus \{\omega\}, F) \xrightarrow{\psi} \mathcal{C}_0(\Gamma, F) \longrightarrow 0$$

is an exact sequence in \mathfrak{M}_E . By b), $\mathcal{C}_0(\mathbb{M} \setminus \Gamma_0^{\mathbb{M}}, F)$ is K-null so by the Topological six-term sequence (Proposition 2.1.8 a)), $K_i(\psi)$ is a group isomorphism. Since Γ is homeomorphic to \mathbb{R} , $K_i(\mathcal{C}_0(\Gamma, F)) \approx K_{i+1}(F)$ by Theorem 3.2.2 a).

e) follows from d) and Proposition 2.4.11. ■

PROPOSITION 3.4.7 *Put*

$$\Gamma' := \Gamma_0^{\mathbb{M}} \cup \Gamma_1^{\mathbb{M}}, \quad \Gamma'' := \Gamma_0^{\mathbb{M}} \cup \Gamma_1^{\mathbb{M}} \cup \Gamma_{-1}^{\mathbb{M}}, \quad \Gamma''' := \Gamma_1^{\mathbb{M}} \cup \Gamma_{-1}^{\mathbb{M}},$$

$$\mathbb{M}' := \mathbb{M} \setminus \Gamma', \quad \mathbb{M}'' := \mathbb{M} \setminus \Gamma'', \quad \mathbb{M}''' := \mathbb{M} \setminus \Gamma'''.$$

Let

$$\varphi' : \mathcal{C}_0(\mathbb{M}', F) \longrightarrow \mathcal{C}(\mathbb{M}, F),$$

$$\varphi'' : \mathcal{C}_0(\mathbb{M}'', F) \longrightarrow \mathcal{C}(\mathbb{M}, F),$$

$$\bar{\varphi}' : \mathcal{C}_0(\mathbb{M}', F) \longrightarrow \mathcal{C}_0(\mathbb{M} \setminus \Gamma_0^{\mathbb{M}}, F),$$

$$\bar{\varphi}'' : \mathcal{C}_0(\mathbb{M}'', F) \longrightarrow \mathcal{C}_0(\mathbb{M} \setminus \Gamma_0^{\mathbb{M}}, F),$$

$$\varphi''' : \mathcal{C}_0(\mathbb{M}', F) \longrightarrow \mathcal{C}_0(\mathbb{M}''', F),$$

$$\lambda' : \mathcal{C}(\Gamma_1^{\mathbb{M}}, F) \longrightarrow \mathcal{C}(\Gamma', F),$$

$$\lambda'' : \mathcal{C}(\Gamma''', F) \longrightarrow \mathcal{C}(\Gamma'', F)$$

$$\lambda''' : \mathcal{C}(\Gamma_0^{\mathbb{M}}, F) \longrightarrow \mathcal{C}(\Gamma'', F),$$

be the inclusion maps,

$$\psi' : \mathcal{C}(\mathbb{M}, F) \longrightarrow \mathcal{C}(\Gamma', F), \quad x \longmapsto x|_{\Gamma'},$$

$$\psi'' : \mathcal{C}(\mathbb{M}, F) \longrightarrow \mathcal{C}(\Gamma'', F), \quad x \longmapsto x|_{\Gamma''},$$

$$\tilde{\psi}' : \mathcal{C}_0(\mathbb{M} \setminus \Gamma_0^{\mathbb{M}}, F) \longrightarrow \mathcal{C}(\Gamma_1^{\mathbb{M}}, F), \quad x \longmapsto x|_{\Gamma_1^{\mathbb{M}}},$$

$$\tilde{\psi}'' : \mathcal{C}_0(\mathbb{M} \setminus \Gamma_0^{\mathbb{M}}, F) \longrightarrow \mathcal{C}(\Gamma''', F), \quad x \longmapsto x|_{\Gamma'''},$$

$$\psi''' : \mathcal{C}_0(\mathbb{M}''', F) \longrightarrow \mathcal{C}(\Gamma_0^{\mathbb{M}}, F), \quad x \longmapsto x|_{\Gamma_0^{\mathbb{M}}},$$

and $\delta'_i, \delta''_i, \bar{\delta}'_i, \bar{\delta}''_i, \delta'''_i$ the index maps associated to the exact sequences in \mathfrak{M}_E

$$0 \longrightarrow \mathcal{C}_0(\mathbb{M}', F) \xrightarrow{\phi'} \mathcal{C}(\mathbb{M}, F) \xrightarrow{\psi'} \mathcal{C}(\Gamma', F) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{C}_0(\mathbb{M}'', F) \xrightarrow{\phi''} \mathcal{C}(\mathbb{M}, F) \xrightarrow{\psi''} \mathcal{C}(\Gamma'', F) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{C}_0(\mathbb{M}', F) \xrightarrow{\phi'} \mathcal{C}(\mathbb{M} \setminus \Gamma_0^{\mathbb{M}}, F) \xrightarrow{\tilde{\psi}'} \mathcal{C}(\Gamma_1^{\mathbb{M}}, F) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{C}_0(\mathbb{M}'', F) \xrightarrow{\bar{\phi}''} \mathcal{C}(\mathbb{M} \setminus \Gamma_0^{\mathbb{M}}, F) \xrightarrow{\tilde{\psi}''} \mathcal{C}(\Gamma''', F) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{C}_0(\mathbb{M}''', F) \xrightarrow{\phi'''} \mathcal{C}_0(\mathbb{M}''', F) \xrightarrow{\psi'''} \mathcal{C}(\Gamma_0^{\mathbb{M}}, F) \longrightarrow 0,$$

respectively.

a) Γ''' is homeomorphic to \mathbb{T} .

b) The maps

$$\bar{\delta}'_i : K_i(\mathcal{C}(\Gamma_1^{\mathbb{M}}, F)) \approx K_i(F) \longrightarrow K_{i+1}(\mathcal{C}_0(\mathbb{M}', F)),$$

$$\bar{\delta}''_i : K_i(\mathcal{C}(\Gamma''', F)) \approx K_i(F) \times K_{i+1}(F) \longrightarrow K_{i+1}(\mathcal{C}_0(\mathbb{M}'', F))$$

are group isomorphisms.

c) If we put $\Phi'_i := K_i(\lambda') \circ (\tilde{\delta}'_i)^{-1}$, $\Phi''_i := K_i(\lambda'') \circ (\tilde{\delta}''_i)^{-1}$ (using b)) then the sequences

$$0 \longrightarrow K_i(\mathcal{C}(\mathbf{IM}, F)) \xrightarrow{K_i(\psi')} K_i(\mathcal{C}(\Gamma', F)) \xrightleftharpoons[\Phi'_i]{\delta'_i} K_{i+1}(\mathcal{C}_0(\mathbf{IM}', F)) \longrightarrow 0,$$

$$0 \longrightarrow K_i(\mathcal{C}(\mathbf{IM}, F)) \xrightarrow{K_i(\psi'')} K_i(\mathcal{C}(\Gamma'', F)) \xrightleftharpoons[\Phi''_i]{\delta''_i} K_{i+1}(\mathcal{C}_0(\mathbf{IM}'', F)) \longrightarrow 0$$

are split exact and the maps

$$K_i(\mathcal{C}(\mathbf{IM}, F)) \times K_{i+1}(\mathcal{C}_0(\mathbf{IM}', F)) \longrightarrow K_i(\mathcal{C}(\Gamma', F)),$$

$$(a, b) \longmapsto K_i(\psi')a + \Phi'_i b,$$

$$K_i(\mathcal{C}(\mathbf{IM}, F)) \times K_{i+1}(\mathcal{C}_0(\mathbf{IM}'', F)) \longrightarrow K_i(\mathcal{C}(\Gamma'', F)),$$

$$(a, b) \longmapsto K_i(\psi'')a + \Phi''_i b$$

are group isomorphisms.

d) $\delta'''_i = 0$ and the sequence

$$0 \longrightarrow K_i(\mathcal{C}_0(\mathbf{IM}'', F)) \xrightarrow{K_i(\varphi''')} K_i(\mathcal{C}_0(\mathbf{IM}''', F))$$

$$K_i(\mathcal{C}_0(\mathbf{IM}''', F)) \xrightarrow{K_i(\psi''')} K_i(\mathcal{C}(\Gamma_0^{\mathbf{IM}}, F)) \longrightarrow 0$$

is exact.

a) is easy to see.

b) By Proposition 3.4.6 b), $\mathcal{C}_0(\mathbf{IM} \setminus \Gamma_0^{\mathbf{IM}}, F)$ is K-null and the assertion follows from a), the Topological six-term sequence (Proposition 2.1.8 b)), and Proposition 3.4.6 a) (and Theorem 3.1.2 a), Theorem 3.2.2 b)).

c) If we put $\Omega_1 := \mathbf{IM}$, $\Omega_2 := \mathbf{IM} \setminus \Gamma_0^{\mathbf{IM}}$, and $\Omega_3 := \mathbf{IM}'$ (respectively $\Omega_3 := \mathbf{IM}''$) then the assertion follows from the Topological triple (Proposition 2.1.11 a)).

d) By the commutativity of the index maps (Axiom 1.2.8), $\delta'''_i = \delta''_i \circ K_i(\lambda''')$. By c), $\text{Im}(\Phi''_i \circ \delta''_i) \subset \text{Im} K_i(\lambda'')$. Since $\text{Im} K_i(\lambda''') = K_i(\mathcal{C}(\Gamma_0^{\mathbf{IM}}, F))$ we get

$$\Phi''_i \circ \delta'''_i = \Phi''_i \circ \delta''_i \circ K_i(\lambda''') = 0.$$

Thus $\delta'''_i = \delta''_i \circ \Phi''_i \circ \delta'''_i = 0$ and the assertion follows from the six-term axiom (Axiom 1.2.7). ■

DEFINITION 3.4.8 We denote by \mathbb{K} the **Klein bottle** obtained from the Möbius band \mathbb{M} by identifying the points $(\alpha, -1)$ and $(\alpha, 1)$ for all $\alpha \in [0, 1]$ and put for every $j \in \{0, 1\}$

$$\Gamma_j^{\mathbb{K}} := \{ (\alpha, j) \in \mathbb{K} \mid \alpha \in [0, 1] \} .$$

PROPOSITION 3.4.9 We put $\mathbb{K}' := \mathbb{K} \setminus \Gamma_0^{\mathbb{K}}$, $\mathbb{K}'' := \mathbb{K} \setminus (\Gamma_0^{\mathbb{K}} \cup \Gamma_1^{\mathbb{K}})$,

$$\psi : \mathcal{C}_0(\mathbb{K}', F) \longrightarrow \mathcal{C}(\Gamma_1^{\mathbb{K}}, F) , \quad x \longmapsto x|_{\Gamma_1^{\mathbb{K}}}$$

and denote by $\varphi : \mathcal{C}_0(\mathbb{K}'', F) \longrightarrow \mathcal{C}_0(\mathbb{K}', F)$ the inclusion map and by δ_i the index maps associated to the exact sequence in \mathfrak{M}_E

$$0 \longrightarrow \mathcal{C}_0(\mathbb{K}'', F) \xrightarrow{\varphi} \mathcal{C}_0(\mathbb{K}', F) \xrightarrow{\psi} \mathcal{C}(\Gamma_1^{\mathbb{K}}, F) \longrightarrow 0 .$$

We use the notation of Proposition 3.4.7 (so $\Gamma_0^{\mathbb{K}} = \Gamma_0^{\mathbb{M}}$ and $\mathbb{K}'' = \mathbb{M}''$).

a) $\Gamma_0^{\mathbb{K}}$ and $\Gamma_1^{\mathbb{K}}$ are homeomorphic to \mathbb{T} .

b) The map

$$(\bar{\delta}_{i+1}'')^{-1} : K_i(\mathcal{C}_0(\mathbb{M}'', F)) \longrightarrow K_{i+1}(\mathcal{C}(\Gamma_1^{\mathbb{K}}, F)) \approx K_i(F) \times K_{i+1}(F)$$

is a group isomorphism.

c) If we identify $K_i(\mathcal{C}(\Gamma_1^{\mathbb{K}}, F))$ with $K_i(F) \times K_{i+1}(F)$ using a) and Theorem 3.2.2 b) and $K_{i+1}(\mathcal{C}_0(\mathbb{K}'', F))$ with $K_i(F) \times K_{i+1}(F)$ using b) then

$$\delta_i : K_i(\mathcal{C}(\Gamma_1^{\mathbb{K}}, F)) \longrightarrow K_{i+1}(\mathcal{C}_0(\mathbb{K}'', F)) , \quad (a, b) \longmapsto (a, 2b) .$$

d) If δ_i is injective then ψ is K -null and $K_i(\mathcal{C}_0(\mathbb{K}', F)) \approx \frac{K_i(F)}{2K_i(F)}$ and if we denote by

$$\Phi_i : K_i(F) \longrightarrow \frac{K_i(F)}{2K_i(F)}$$

the quotient map then

$$K_i(\varphi) : K_i(\mathcal{C}_0(\mathbb{K}'', F)) \longrightarrow K_i(\mathcal{C}_0(\mathbb{K}', F)) , \quad (a, b) \longmapsto \Phi_i b .$$

a) is easy to see.

b) follows from Proposition 3.4.7 b).

c) We denote by

$$\vartheta : \mathbb{M} \setminus \Gamma_0^{\mathbb{M}} \longrightarrow \mathbb{K}'$$

the covering map, by

$$\vartheta' : \Gamma''' \longrightarrow \Gamma_1^{\mathbb{K}}$$

the map defined by ϑ , and put

$$\phi : \mathcal{C}_0(\mathbb{K}', F) \longrightarrow \mathcal{C}_0(\mathbb{M} \setminus \Gamma_0^{\mathbb{M}}, F), \quad x \longmapsto x \circ \vartheta,$$

$$\phi' : \mathcal{C}_0(\Gamma_1^{\mathbb{K}}, F) \longrightarrow \mathcal{C}_0(\Gamma''', F), \quad x \longmapsto x \circ \vartheta'.$$

With the identifications of Γ''' and $\Gamma_1^{\mathbb{K}}$ with \mathbb{T} (by a) and Proposition 3.4.7 a)),

$$\vartheta' : \Gamma''' \longrightarrow \Gamma_1^{\mathbb{K}}, \quad z \longmapsto z^2.$$

By the commutativity of the index maps (Axiom 1.2.8) the diagrams

$$\begin{array}{ccccc} K_i(\mathcal{C}_0(\mathbb{M}'', F)) & \xrightarrow{K_i(\phi)} & K_i(\mathcal{C}_0(\mathbb{K}', F)) & \xrightarrow{K_i(\psi)} & K_i(\mathcal{C}(\Gamma_1^{\mathbb{K}}, F)) \\ \downarrow = & & \downarrow K_i(\phi) & & \downarrow K_i(\phi') \\ K_i(\mathcal{C}_0(\mathbb{M}'', F)) & \xrightarrow{K_i(\bar{\phi}'')} & K_i(\mathcal{C}_0(\mathbb{M}''', F)) & \xrightarrow{K_i(\bar{\psi}'')} & K_i(\mathcal{C}(\Gamma''', F)) \\ & & & & \\ K_i(\mathcal{C}(\Gamma_1^{\mathbb{K}}, F)) & \xrightarrow{\delta_i} & K_{i+1}(\mathcal{C}_0(\mathbb{M}'', F)) & & \\ \downarrow K_i(\phi') & & \downarrow = & & \\ K_i(\mathcal{C}(\Gamma''', F)) & \xrightarrow{\bar{\delta}_i''} & K_{i+1}(\mathcal{C}_0(\mathbb{M}'', F)) & & \end{array}$$

are commutative. By Proposition 3.3.4 c),

$$K_i(\phi') : K_i(\mathcal{C}(\Gamma_1^{\mathbb{K}}, F)) \longrightarrow K_i(\mathcal{C}(\Gamma''', F)), \quad (a, b) \longmapsto (a, 2b).$$

By b),

$$\delta_i : K_i(\mathcal{C}(\Gamma_1^{\mathbb{K}}, F)) \longrightarrow K_{i+1}(\mathcal{C}_0(\mathbb{K}'', F)), \quad (a, b) \longmapsto (a, 2b).$$

d) By the six-term axiom (Axiom 1.2.7), ψ is K-null. The other assertions follow from c) and the six-term axiom (Axiom 1.2.7). ■

3.5 Pasting Locally Compact Spaces

PROPOSITION 3.5.1 *Let Ω_1, Ω_2 be locally compact spaces, Γ_1 and Γ_2 closed sets of Ω_1 and Ω_2 , respectively, $\vartheta : \Gamma_1 \longrightarrow \Gamma_2$ a homeomorphism, Ω' the topological sum of $\Omega_1 \setminus \Gamma_1$ and $\Omega_2 \setminus \Gamma_2$, Ω the locally compact space obtained from the topological sum of Ω_1 and Ω_2 by identifying the points ω and $\vartheta(\omega)$ for all $\omega \in \Gamma_1$, Γ the closed set of Ω corresponding to the identified Γ_1 and Γ_2 (so $\Omega \setminus \Gamma = \Omega'$), $\varphi : \mathcal{C}_0(\Omega \setminus \Gamma, F) \longrightarrow \mathcal{C}_0(\Omega, F)$ the inclusion map,*

$$\psi : \mathcal{C}_0(\Omega, F) \longrightarrow \mathcal{C}_0(\Gamma, F), \quad x \longmapsto x|_{\Gamma},$$

and δ_i the index maps associated to the exact sequence in \mathfrak{M}_E

$$0 \longrightarrow \mathcal{C}_0(\Omega \setminus \Gamma, F) \xrightarrow{\varphi} \mathcal{C}_0(\Omega, F) \xrightarrow{\psi} \mathcal{C}_0(\Gamma, F) \longrightarrow 0.$$

Let $J := \{1, 2\}$ and for every $j \in J$ let

$$\varphi_j : \mathcal{C}_0(\Omega_j \setminus \Gamma_j, F) \longrightarrow \mathcal{C}_0(\Omega_j, F),$$

$$\varphi'_j : \mathcal{C}_0(\Omega_j \setminus \Gamma_j, F) \longrightarrow \mathcal{C}_0(\Omega', F),$$

$$\varphi''_j : \mathcal{C}_0(\Omega_j \setminus \Gamma_j, F) \longrightarrow \mathcal{C}_0(\Omega, F)$$

be the inclusion maps,

$$\psi_j : \mathcal{C}_0(\Omega_j, F) \longrightarrow \mathcal{C}_0(\Gamma_j, F), \quad x \longmapsto x|_{\Gamma_j},$$

$$\psi'_j : \mathcal{C}_0(\Omega', F) \longrightarrow \mathcal{C}_0(\Omega_j \setminus \Gamma_j, F), \quad x \longmapsto x|_{(\Omega_j \setminus \Gamma_j)},$$

and $\delta_{j,i}$ the index maps associated to the exact sequence in \mathfrak{M}_E

$$0 \longrightarrow \mathcal{C}_0(\Omega_j \setminus \Gamma_j, F) \xrightarrow{\varphi_j} \mathcal{C}_0(\Omega_j, F) \xrightarrow{\psi_j} \mathcal{C}_0(\Gamma_j, F) \longrightarrow 0.$$

a) $\delta_{j,i} = K_{i+1}(\psi'_j) \circ \delta_i$ for every $j \in J$ and

$$\delta_i = K_{i+1}(\varphi'_1) \circ \delta_{1,i} + K_{i+1}(\varphi'_2) \circ \delta_{2,i}.$$

b) Assume $\mathcal{C}_0(\Omega_1, F)$ K -null.

$b_1)$ $\delta_{1,i} : K_i(\mathcal{C}_0(\Gamma_1, F)) \longrightarrow K_{i+1}(\mathcal{C}_0(\Omega_1 \setminus \Gamma_1, F))$ is a group isomorphism.

$b_2)$ δ_i is injective.

$b_3)$ ψ is K -null.

$b_4)$ $K_i(\varphi_2'') : K_i(\mathcal{C}_0(\Omega_2 \setminus \Gamma_2, F)) \longrightarrow K_i(\mathcal{C}_0(\Omega, F))$ is a group isomorphism.

$b_5)$ If we put

$$\Phi_i := K_i(\varphi_2') \circ K_i(\varphi_2'')^{-1} : K_i(\mathcal{C}_0(\Omega, F)) \longrightarrow K_i(\mathcal{C}_0(\Omega', F))$$

then the map

$$K_{i+1}(\mathcal{C}_0(\Gamma, F)) \times K_i(\mathcal{C}_0(\Omega, F)) \longrightarrow K_i(\mathcal{C}_0(\Omega', F)),$$

$$(a, b) \longmapsto \delta_{i+1}a + \Phi_i b$$

is a group isomorphism.

$b_6)$ If also $\mathcal{C}_0(\Omega_2, F)$ is K -null then

$$K_i(\mathcal{C}_0(\Omega, F)) \approx K_{i+1}(\mathcal{C}_0(\Gamma, F)),$$

$$K_i(\mathcal{C}_0(\Omega', F)) \approx K_{i+1}(\mathcal{C}_0(\Gamma, F))^2.$$

a) follows from Proposition 2.3.7 a), since ψ_j'' of this Proposition is the identity map in the present case.

$b_1)$ follows from the Topological six-term sequence (Proposition 2.1.8 a)).

$b_2)$ Let $a \in K_i(\mathcal{C}_0(\Gamma, F))$ such that $\delta_i a = 0$. By a), $\delta_{1,i}a = K_{i+1}(\varphi_1')\delta_i a = 0$ and by $b_1)$, $a = 0$.

$b_3)$ follows from $b_2)$ and the six-term axiom (Axiom 1.2.7).

$b_4)$ and $b_5)$ follow from $b_3)$ and Proposition 2.3.7 $c_1)$, $c_2)$.

$b_6)$ follows from $b_1)$, $b_4)$, and the Product Theorem (Proposition 2.3.1 a)). ■

COROLLARY 3.5.2 *Let Γ be a locally compact space, $(\Omega_j)_{j \in J}$ a nonempty finite family of locally compact spaces such that $\mathcal{C}_0(\Omega_j, F)$ is K -null for every $j \in J$, and for every $j \in J$ let Γ_j be a closed set of Ω_j and $\vartheta_j : \Gamma \longrightarrow \Gamma_j$ a homeomorphism. Let Ω' the topological sum of the family $(\Omega_j \setminus \Gamma_j)_{j \in J}$, and Ω the locally compact space obtained*

from the topological sum of the family $(\Omega_j)_{j \in J}$ by identifying for every $\omega \in \Gamma$ all the points $\vartheta_j(\omega)$ ($j \in J$). Then

$$K_i(\mathcal{C}_0(\Omega, F)) \approx K_{i+1}(\mathcal{C}_0(\Gamma, F))^{\text{Card } J-1},$$

$$K_i(\mathcal{C}_0(\Omega', F)) \approx K_i(\mathcal{C}_0(\Omega, F)) \times K_{i+1}(\mathcal{C}_0(\Gamma, F)) \approx K_{i+1}(\mathcal{C}_0(\Gamma, F))^{\text{Card } J}.$$

We prove the Corollary by induction with respect to $\text{Card } J$. For $\text{Card } J \in \{1, 2\}$ the assertion follows from Proposition 3.5.1 $b_1), b_5), b_6)$. Let $k \in J$, assume the assertion holds for $J' := J \setminus \{k\}$, and denote by Ω'' the topological sum of the family $(\Omega_j \setminus \Gamma_j)_{j \in J'}$. By Proposition 3.5.1 $b_4), b_5)$ and the induction hypothesis,

$$K_i(\mathcal{C}_0(\Omega, F)) \approx K_i(\mathcal{C}_0(\Omega'', F)) \approx K_{i+1}(\mathcal{C}_0(\Gamma, F))^{\text{Card } J-1},$$

$$K_i(\mathcal{C}_0(\Omega', F)) \approx K_{i+1}(\mathcal{C}_0(\Gamma, F)) \times K_i(\mathcal{C}_0(\Omega'', F)) \approx K_{i+1}(\mathcal{C}_0(\Gamma, F))^J. \quad \blacksquare$$

COROLLARY 3.5.3 *Let $m, n \in \mathbb{N}$,*

$$\Gamma_+ := \{ \alpha \in \mathbb{B}_n \mid \|\alpha\| = 1, \alpha_n > 0 \}, \Gamma_- := \{ \alpha \in \mathbb{B}_n \mid \|\alpha\| = 1, \alpha_n \leq 0 \},$$

and Ω the locally compact space obtained from the topological sum of the family $(\mathbb{B}_n \setminus \Gamma_-)_{j \in \mathbb{N}_m}$ by identifying all the Γ_+ . Then

$$K_i(\mathcal{C}_0(\Omega, F)) \approx K_{i+n}(F)^{m-1}.$$

By Proposition 2.4.1, $\mathcal{C}_0(\mathbb{B}_n \setminus \Gamma_-, F)$ is null-homotopic and so K -null. For $n > 1$, Γ_+ is homeomorphic to \mathbb{R}^{n-1} so by Theorem 3.2.2 a),

$$K_i(\mathcal{C}_0(\Gamma_+, F)) \approx K_{i+n-1}(F)$$

and this relation obviously holds also for $n = 1$. Then by Corollary 3.5.2,

$$K_i(\mathcal{C}_0(\Omega, F)) \approx K_{i+n}(F)^{m-1}. \quad \blacksquare$$

Remark. The above result can be deduced also from Example 2.4.9 by using Proposition 1.5.11 d).

COROLLARY 3.5.4 *Let Ω', Ω'' be locally compact spaces, $\omega' \in \Omega'$, $\omega'' \in \Omega''$, and Ω the locally compact space obtained from the topological sum of Ω' and Ω'' by identifying ω' and ω'' . If $\mathcal{C}_0(\Omega'', F)$ is K -null then*

$$K_i(\mathcal{C}_0(\Omega, F)) \approx K_i(\mathcal{C}_0(\Omega' \setminus \{\omega'\}, F)) .$$

The assertion follows from Proposition 3.5.1 b_4). ■

PROPOSITION 3.5.5 *Let Ω', Ω'' be compact spaces, $\omega' \in \Omega'$, $\omega'' \in \Omega''$, and Ω the compact space obtained by identifying the points ω' and ω'' in the topological sum of Ω' and Ω'' . Then*

$$K_i(\mathcal{C}(\Omega, F)) \approx K_i(\mathcal{C}_0(\Omega \setminus \Omega', F)) \times K_i(\mathcal{C}(\Omega', F)) .$$

Let $\phi : \mathcal{C}_0(\Omega \setminus \Omega', F) \longrightarrow \mathcal{C}(\Omega, F)$ be the inclusion map and

$$\psi : \mathcal{C}(\Omega, F) \longrightarrow \mathcal{C}(\Omega', F) , \quad x \longmapsto x|_{\Omega'} .$$

We put for every $x \in \mathcal{C}(\Omega', F)$,

$$\lambda x : \Omega \longrightarrow F , \quad \omega \longmapsto \begin{cases} x(\omega) & \text{if } \omega \in \Omega' \\ x(\omega_0) & \text{if } \omega \in \Omega'' \end{cases} ,$$

where $\omega_0 \in \Omega$ denotes the point corresponding to the identified points ω' and ω'' . Then

$$0 \longrightarrow \mathcal{C}_0(\Omega \setminus \Omega', F) \xrightarrow{\phi} \mathcal{C}(\Omega, F) \xrightarrow[\lambda]{\psi} \mathcal{C}(\Omega', F) \longrightarrow 0$$

is a split exact sequence in \mathfrak{M}_E and the assertion follows from the split exact axiom (Axiom 1.2.3). ■

PROPOSITION 3.5.6 *Let $(\Omega_j)_{j \in \mathbb{N}_n}$ be a family of compact spaces and for every $j \in \mathbb{N}_n$ let ω_j, ω'_j be distinct points of Ω_j . If Ω denotes the compact space obtained from the topological sum of the family $(\Omega_j)_{j \in \mathbb{N}_n}$ by identifying ω'_j with ω_{j+1} for all $j \in \mathbb{N}_{n-1}$ then*

$$K_i(\mathcal{C}(\Omega, F)) \approx K_i(F) \times \prod_{j=1}^n K_i(\mathcal{C}_0(\Omega_j \setminus \{\omega_j\}, F)) .$$

If $(k_j)_{j \in \mathbb{N}_n}$ is a family in \mathbb{N} , $\Omega_j = \mathbf{S}_{k_j}$ for every $j \in \mathbb{N}_n$, and

$$p := \text{Card} \{ j \in \mathbb{N}_n \mid k_j \text{ is even} \} , \quad q := \text{Card} \{ j \in \mathbb{N}_n \mid k_j \text{ is odd} \}$$

then

$$K_i(\mathcal{C}(\Omega, F)) \approx K_i(F)^{p+1} \times K_{i+1}(F)^q.$$

We put $\bar{\Omega}_n := \Omega$ and prove the assertion by induction with respect to $n \in \mathbb{N}$. For $n = 1$ the assertion follows from the Alexandroff K-theorem (Theorem 2.2.1 a)). Assume the assertion holds for an $n \in \mathbb{N}$. By Proposition 3.5.5 and the induction hypothesis,

$$\begin{aligned} K_i(\mathcal{C}(\bar{\Omega}_{n+1}, F)) &\approx K_i(\mathcal{C}_0(\bar{\Omega}_{n+1} \setminus \bar{\Omega}_n, F)) \times K_i(\mathcal{C}(\bar{\Omega}_n, F)) \approx \\ &\approx K_i(\mathcal{C}_0(\Omega_{n+1} \setminus \{\omega_{n+1}\}, F)) \times K_i(\mathcal{C}(\bar{\Omega}_n, F)) \approx \\ &\approx K_i(\mathcal{C}_0(\Omega_{n+1} \setminus \{\omega_{n+1}\}, F)) \times K_i(F) \times \prod_{j=1}^n K_i(\mathcal{C}_0(\Omega_j \setminus \{\omega_j\}, F)) \approx \\ &\approx K_i(F) \times \prod_{j=1}^{n+1} K_i(\mathcal{C}_0(\Omega_j \setminus \{\omega_j\}, F)), \end{aligned}$$

which finishes the inductive proof. The last assertion follows now from Theorem 3.2.2 a), since $\mathbf{S}_{k_j} \setminus \{\omega_j\}$ is homeomorphic to \mathbb{R}^{k_j} . ■

PROPOSITION 3.5.7 *Let Ω_1, Ω_2 be locally compact spaces such that the $E-C^*-$ algebra $\mathcal{C}_0(\Omega_2, F)$ is K -null, Γ a compact set of Ω_1 , and $\vartheta : \Gamma \longrightarrow \Omega_2$ a continuous map. We denote by Ω the locally compact space obtained from the topological sum of Ω_1 and Ω_2 by identifying the points ω and $\vartheta(\omega)$ for all $\omega \in \Gamma$.*

a) If

$$\varphi : \mathcal{C}_0(\Omega_1 \setminus \Gamma, F) \longrightarrow \mathcal{C}_0(\Omega, F)$$

denotes the inclusion map then

$$K_i(\varphi) : K_i(\mathcal{C}_0(\Omega_1 \setminus \Gamma, F)) \longrightarrow K_i(\mathcal{C}_0(\Omega, F))$$

is a group isomorphism. If in addition $\Omega \in \Upsilon$ or $\Omega_1 \setminus \Gamma \in \Upsilon$ then

$$\Omega, \Omega_1 \setminus \Gamma \in \Upsilon, \quad p(\Omega) = p(\Omega_1 \setminus \Gamma), \quad q(\Omega) = q(\Omega_1 \setminus \Gamma), \quad \Omega_\Upsilon = (\Omega_1 \setminus \Gamma)_\Upsilon.$$

b) If Ω^* denotes the Alexandroff compactification of Ω then

$$K_i(\mathcal{C}(\Omega^*, F)) \approx K_i(F) \times K_i(\mathcal{C}_0(\Omega_1 \setminus \Gamma, F)).$$

a) If we put

$$\psi : \mathcal{C}_0(\Omega, F) \longrightarrow \mathcal{C}_0(\Omega_2, F), \quad x \longmapsto x|_{\Omega_2}$$

then

$$0 \longrightarrow \mathcal{C}_0(\Omega_1 \setminus \Gamma, F) \xrightarrow{\varphi} \mathcal{C}_0(\Omega, F) \xrightarrow{\psi} \mathcal{C}_0(\Omega_2, F) \longrightarrow 0$$

is an exact sequence in \mathfrak{M}_E . Since $\mathcal{C}_0(\Omega_2, F)$ is K-null, the assertion follows from the Topological six-term sequence (Proposition 2.1.8 c)).

b) follows from a) and Alexandroff's K-theorem (Theorem 2.2.1 a)). ■

COROLLARY 3.5.8 *Let $(\Omega_j)_{j \in J}$ be a finite family of locally compact spaces, $\omega_j \in \Omega_j$ for all $j \in J$, and Ω the locally compact space obtained from the topological sum of the family $(\Omega_j)_{j \in J}$ by identifying the points ω_j for all $j \in J$.*

a) *If there is a $j_0 \in J$ such that $\mathcal{C}_0(\Omega_{j_0}, F)$ is K-null then*

$$K_i(\mathcal{C}_0(\Omega, F)) \approx \prod_{j \in J \setminus \{j_0\}} K_i(\mathcal{C}_0(\Omega_j \setminus \{\omega_j\}, F)).$$

b) *If $\Omega_j := [0, 1[$ for all $j \in J$ and $n := \text{Card } J$ then*

$$K_i(\mathcal{C}_0(\Omega, F)) \approx K_{i+1}(F)^{n-1}.$$

c) *Let $j_0 \in J$ and $\Omega_{j_0} := [0, 1[$. If $(k_j)_{j \in J \setminus \{j_0\}}$ is a family in \mathbb{N} ,*

$$p := \text{Card} \{ j \in J \setminus \{j_0\} \mid k_j \text{ is even} \},$$

$$q := \text{Card} \{ j \in J \setminus \{j_0\} \mid k_j \text{ is odd} \},$$

and $\Omega_j := \mathbf{S}_{k_j}$ for every $j \in J \setminus j_0$ then

$$K_i(\mathcal{C}_0(\Omega, F)) \approx K_i(F)^p \times K_{i+1}(F)^q.$$

a) Let Ω' be the locally compact space obtained from the topological sum of the family $(\Omega_j)_{j \in J \setminus \{j_0\}}$ by identifying the points ω_j for all $j \in J \setminus \{j_0\}$ and let $\bar{\omega}$ denote the point obtained by this identification. If we replace in Proposition 3.5.7 Ω_1 by Ω' , Ω_2 by Ω_{j_0} , Γ by $\bar{\omega}$, and take $\vartheta(\bar{\omega}) := \omega_{j_0}$ then we get

$$K_i(\mathcal{C}_0(\Omega, F)) \approx K_i(\mathcal{C}_0(\Omega' \setminus \{\bar{\omega}\}, F)).$$

$\Omega' \setminus \{\bar{\omega}\}$ is the topological sum of the family $(\Omega_j \setminus \{\omega_j\})_{j \in J \setminus \{j_0\}}$ so by the Product Theorem (Proposition 2.3.1 a)),

$$K_i(\mathcal{C}_0(\Omega' \setminus \{\bar{\omega}\}, F)) \approx \prod_{j \in J \setminus \{j_0\}} K_i(\mathcal{C}_0(\Omega_j \setminus \{\omega_j\}, F)) .$$

b) follows immediately from a) since $\mathcal{C}_0([0, 1[, F)$ is K-null and

$$K_i(\mathcal{C}_0([0, 1[\setminus \{\omega\}, F)) \approx K_{i+1}(F)$$

for all $\omega \in [0, 1[$.

c) For $j \in J \setminus \{j_0\}$, $\mathbf{S}_{k_j} \setminus \{\omega_j\}$ is homeomorphic to \mathbb{R}^{k_j} and so by Theorem 3.2.2 a), $K_i(\mathcal{C}_0(\mathbf{S}_{k_j} \setminus \{\omega_j\}, F)) \approx K_{i+k_j}(F)$. Since $\mathcal{C}_0([0, 1[, F)$ is K-null, we get from a),

$$K_i(\mathcal{C}_0(\Omega, F)) \approx K_i(F)^p \times K_{i+1}(F)^q . \quad \blacksquare$$

COROLLARY 3.5.9 *Let J_1, J_2, J_3 be pairwise disjoint finite sets and let Ω be the locally compact space (the **graph**) obtained from the topological sum of $[0, 1] \times J_1$, $[0, 1] \times J_2$, and $[0, 1] \times J_3$ by identifying some of the points of the set*

$$\{ (0, j) \mid j \in J_1 \cup J_2 \} \cup \{ (1, j) \mid j \in J_1 \} .$$

If s denotes the number of compact connected components of Ω and r_0 and r_1 denote the number of vertices and chords of the graph Ω , respectively, then

$$K_i(\mathcal{C}_0(\Omega, F)) \approx K_i(F)^s \times K_{i+1}(F)^{s+r_1-r_0} .$$

By the Product Theorem (Proposition 2.3.1 a)), we may assume Ω connected.

Assume first there is a $j \in J_3$ such that Ω contains $]0, 1[\times \{j\}$. Since Ω is connected, $\Omega =]0, 1[\times \{j\}$. Thus Ω is homeomorphic to \mathbb{R} , $r_1 - r_0 = 1$, and the assertion follows from Theorem 3.2.2 a).

Assume now there is a $j \in J_2$ such that Ω contains $[0, 1] \times \{j\}$. By Proposition 3.5.7 a),

$$K_i(\mathcal{C}_0(\Omega, F)) \approx K_i(\mathcal{C}_0(\Omega \setminus ([0, 1] \times \{j\}), F)) .$$

Ω and $\Omega \setminus ([0, 1] \times \{j\})$ have the same $r_1 - r_0$, so we may replace Ω by $\Omega \setminus ([0, 1] \times \{j\})$. Repeating the operation, we obtain finally a locally compact space, which is the

topological sum of a finite family $(]0, 1[)_{j \in J}$, and in this case the assertion follows from the Product Theorem (Proposition 2.3.1 a)) and Theorem 3.2.2 a).

Finally assume Ω compact. Then there is a $j \in J_1$ such that Ω contains $[0, 1] \times \{j\}$. By the above and by Alexandroff's K-theorem

$$K_i(\mathcal{C}(\Omega, F)) \approx K_i(F) \times K_i(\mathcal{C}_0(\Omega \setminus \{(1, j)\}, F)) .$$

If s', r'_0, r'_1 denote the corresponding numbers associated to $\Omega \setminus \{(1, j)\}$ then $s' = 0$, $r'_0 = r_0 - 1$, and $r'_1 = r_1$. All the connected components of $\Omega \setminus \{(1, j)\}$ satisfy the condition of the above paragraphs, so

$$K_i(\mathcal{C}_0(\Omega \setminus \{(1, j)\}, F)) \approx K_{i+1}(F)^{r'_1 - r'_0} \approx K_{i+1}(F)^{1 + r_1 - r_0} ,$$

$$K_i(\mathcal{C}(\Omega, F)) \approx K_i(F) \times K_{i+1}(F)^{1 + r_1 - r_0} . \quad \blacksquare$$

COROLLARY 3.5.10 *If Ω is a compact graph contained in \mathbb{B}_n then*

$$K_i(\mathcal{C}_0(\mathbb{B}_n \setminus \Omega, F)) \approx K_i(F)^{s - r_0 + r_1} \times K_{i+1}(F)^{s-1} ,$$

where s denotes the number of connected components of Ω and r_0 and r_1 the number of vertices and chords of Ω , respectively.

Let ω be a vertex of Ω . By Corollary 3.5.9 and Corollary 2.4.4 a),

$$K_i(\mathcal{C}_0(\Omega \setminus \{\omega\}, F)) \approx K_i(F)^{s-1} \times K_{i+1}(F)^{s - r_0 + r_1}$$

and by Theorem 3.1.2 b),

$$K_i(\mathcal{C}_0(\mathbb{B}_n \setminus \Omega, F)) \approx K_{i+1}(\mathcal{C}_0(\Omega \setminus \{\omega\}, F)) \approx K_i(F)^{s - r_0 + r_1} \times K_{i+1}(F)^{s-1} . \quad \blacksquare$$

EXAMPLE 3.5.11 *Let $n \in \mathbb{N}$, Γ a closed set of \mathbf{S}_n , $\emptyset \neq \Gamma \neq \mathbf{S}_n$, $\omega \in \Gamma$, Γ' the compact space obtained from $\Gamma \times [0, 1]$ by identifying the points of $\Gamma \times 0$, and Ω the compact space obtained from the topological sum of \mathbf{S}_n and Γ' by identifying the points of $\Gamma \subset \mathbf{S}_n$ with the points of $\Gamma \times \{1\} \subset \Gamma'$.*

$$a) \quad K_i(\mathcal{C}(\Omega, F)) \approx K_i(F) \times K_{i+n}(F) \times K_{i+1}(\mathcal{C}_0(\Gamma \setminus \{\omega\}, F)) .$$

b) If Γ is finite then

$$K_i(\mathcal{C}(\Omega, F)) \approx K_i(F) \times K_{i+n}(F) \times K_{i+1}(F)^{\text{Card}\Gamma-1}.$$

c) If Γ is a graph then

$$K_i(\mathcal{C}(\Omega, F)) \approx K_i(F)^{1+s+r_1-r_0} \times K_{i+n}(F) \times K_{i+1}(F)^{s-1},$$

where s denotes the number of connected components of Ω and r_0 and r_1 denote the number of vertices and chords of the graph Γ , respectively,

a) By Theorem 3.2.2 e_1),

$$K_i(\mathcal{C}_0(\mathbf{S}_n \setminus \Gamma, F)) \approx K_{i+n}(F) \times K_{i+1}(\mathcal{C}_0(\Gamma \setminus \{\omega\}, F)).$$

By Proposition 2.4.1, $\mathcal{C}_0(\Gamma \setminus \{0\}, F)$ is K-null, where 0 is the point obtained from the identification of the points of $\Gamma \times \{0\}$. By Proposition 3.5.7 a),

$$K_i(\mathcal{C}_0(\Omega \setminus \{0\}, F)) \approx K_i(\mathcal{C}_0(\mathbf{S}_n \setminus \Gamma, F)),$$

so by Alexandroff's K-theorem (Theorem 2.2.1 a)),

$$K_i(\mathcal{C}(\Omega, F)) \approx K_i(F) \times K_{i+n}(F) \times K_{i+1}(\mathcal{C}_0(\Gamma \setminus \{\omega\}, F)).$$

b) follows from a) and the Product Theorem (Proposition 2.3.1 a)).

c) By Corollary 3.5.9 and Alexandroff's K-theorem (Theorem 2.2.1 a)),

$$K_i(\mathcal{C}(\Gamma, F)) \approx K_i(F)^s \times K_{i+1}(F)^{s+r_1-r_0},$$

$$K_i(\mathcal{C}_0(\Gamma \setminus \{\omega\}, F)) \approx K_i(F)^{s-1} \times K_{i+1}(F)^{s+r_1-r_0},$$

so by a),

$$K_i(\mathcal{C}(\Omega, F)) \approx K_i(F)^{1+s+r_1-r_0} \times K_{i+n}(F) \times K_{i+1}(F)^{s-1}. \quad \blacksquare$$

PROPOSITION 3.5.12 Let $(p_j)_{j \in J}$ be a finite family in \mathbb{N} , ($J \neq \emptyset$), and for every $j \in J$ put $\Omega_j := \mathbf{S}_{p_j}$. Let Ω' be the topological sum of the family $(\Omega_j)_{j \in J}$, $(\Gamma_k)_{k \in K}$ a finite family of pairwise disjoint nonempty finite subsets of Ω' , $\Gamma := \bigcup_{k \in K} \Gamma_k$, and Ω the compact space obtained from Ω' by identifying for every $k \in K$ the points of Γ_k . If Ω is connected then

$$K_i(\mathcal{C}(\Omega, F)) \approx K_i(F) \times K_{i+1}(F)^{\text{Card}\Gamma - \text{Card}J - \text{Card}K + 1} \times \prod_{j \in J} K_{i+p_j}(F).$$

If $K = \emptyset$, since Ω is connected, J is a one-point set and the assertion holds by Theorem 3.2.2 b). Thus we may assume $K = \mathbb{N}_n$ for some $n \in \mathbb{N}$. Take $k_1 \in K$ and put $J_1 := \{j \in J \mid \Omega_j \cap \Gamma_{k_1} \neq \emptyset\}$. We define recursively an injective family $(k_m)_{m \in \mathbb{N}_n}$ in K and an increasing family $(J_m)_{m \in \mathbb{N}_n}$ of subsets of J in the following way. Let $m \in \mathbb{N}_n$, $m > 1$, and assume the families were defined up to $m-1$. Since Ω is connected there is a $k_m \in K \setminus \{k_q \mid q \in \mathbb{N}_{m-1}\}$ such that $\Gamma_{k_m} \cap J_{m-1} \neq \emptyset$. We put

$$J_m := \left\{ j \in J \mid \Omega_j \cap \left(\bigcup_{q=1}^m \Gamma_{k_q} \right) \neq \emptyset \right\}.$$

It is easy to prove by induction with respect to $m \in \mathbb{N}_n$ that

$$\text{Card} \left(\bigcup_{q=1}^m \Gamma_{k_q} \right) - \text{Card} J_m - m + 1 \geq 0$$

for every $m \in \mathbb{N}_n$. In particular,

$$\text{Card} \Gamma - \text{Card} J - \text{Card} K + 1 \geq 0.$$

For every $j \in J$, by Proposition 2.4.11 and Theorem 3.2.2 a),

$$K_i(\mathcal{C}_0(\Omega_j \setminus \Gamma, F)) \approx K_{i+1}(F)^{\text{Card}(\Gamma \cap \Omega_j) - 1} \times K_{i+p_j}(F)$$

so that by the Product Theorem (Proposition 2.3.1 a)),

$$K_i(\mathcal{C}_0(\Omega' \setminus \Gamma, F)) \approx K_{i+1}(F)^{\text{Card} \Gamma - \text{Card} J} \times \prod_{j \in J} K_{i+p_j}(F).$$

For every $k \in K$ let ω_k be the point of Ω corresponding to the unified points of Γ_k and put $\Delta := \{\omega_k \mid k \in K\}$. Then by Proposition 2.4.11,

$$K_i(\mathcal{C}_0(\Omega \setminus \Delta, F)) \approx K_i(\mathcal{C}_0(\Omega \setminus \{\omega_{k_0}\}, F)) \times K_{i+1}(F)^{\text{Card} K - 1},$$

where $k_0 \in K$. By the above and by Alexandroff's K-theorem, since $\Omega \setminus \Delta = \Omega' \setminus \Gamma$,

$$\begin{aligned} & K_i(\mathcal{C}(\Omega, F)) \times K_{i+1}(F)^{\text{Card} K - 1} \approx \\ & \approx K_i(F) \times K_i(\mathcal{C}_0(\Omega \setminus \{\omega_{k_0}\}, F)) \times K_{i+1}(F)^{\text{Card} K - 1} \approx \\ & \approx K_i(F) \times K_i(\mathcal{C}_0(\Omega \setminus \Delta, F)) \approx K_i(F) \times K_i(\mathcal{C}_0(\Omega' \setminus \Gamma, F)) \approx \\ & \approx K_i(F) \times K_{i+1}(F)^{\text{Card} \Gamma - \text{Card} J - \text{Card} K + 1} \times K_{i+1}(F)^{\text{Card} K - 1} \times \prod_{j \in J} K_{i+p_j}(F), \\ & K_i(\mathcal{C}(\Omega, F)) \approx K_i(F) \times K_{i+1}(F)^{\text{Card} \Gamma - \text{Card} J - \text{Card} K + 1} \times \prod_{j \in J} K_{i+p_j}(F). \quad \blacksquare \end{aligned}$$

COROLLARY 3.5.13 *Let $(p_j)_{j \in \mathbb{N}_n}$ be a family in \mathbb{N} and for every $j \in \mathbb{N}_n$ put $\Omega_j := \mathbf{S}_{p_j}$. For every $j \in \mathbb{N}_n$ let Γ_j and Γ'_j be disjoint nonempty finite subsets of Ω_j such that $k_j := \text{Card} \Gamma'_j = \text{Card} \Gamma_{j+1}$ for every $j \in \mathbb{N}_{n-1}$. We denote by Ω the compact space obtained from the topological sum of the family $(\Omega_j)_{j \in \mathbb{N}_n}$ by identifying in a bijective way Γ'_j with Γ_{j+1} for all $j \in \mathbb{N}_{n-1}$. Then*

$$K_i(\mathcal{C}(\Omega, F)) \approx K_i(F) \times K_{i+1}(F)^{\sum_{j=1}^{n-1} (k_j-1)} \times \prod_{j=1}^n K_{i+p_j}(F). \quad \blacksquare$$

PROPOSITION 3.5.14 *Let Ω_1, Ω_2 be locally compact spaces and for every $j \in \{1, 2\}$ let Γ_j be a compact set of Ω_j and $\vartheta_j : \mathbb{B}_n \rightarrow \Gamma_j$ a homeomorphism such that $\Delta_j := \vartheta_j(\mathbb{B}_n \setminus \mathbf{S}_{n-1})$ is an open set of Ω_j . We denote by Ω the locally compact space obtained from the topological sum of $\Omega_1 \setminus \Delta_1$ and $\Omega_2 \setminus \Delta_2$ by identifying the points $\vartheta_1(\omega)$ and $\vartheta_2(\omega)$ for all $\omega \in \mathbf{S}_{n-1}$. Then for every $\omega \in \mathbf{S}_{n-1}$,*

$$\begin{aligned} K_i(\mathcal{C}_0(\Omega \setminus \{\vartheta_1(\omega)\}, F)) &\approx \\ &\approx K_i(\mathcal{C}_0(\Omega_1 \setminus \Gamma_1, F)) \times K_i(\mathcal{C}_0(\Omega_2 \setminus \Gamma_2, F)) \times K_{i+n-1}(F). \end{aligned}$$

We use the notation of the topological triple (Proposition 2.1.11), which we mark with a prime in order to distinguish them from the present notation. We put $\Omega'_2 := \Omega \setminus \{\vartheta_1(\omega)\}$ and take as Ω'_3 the topological sum of $\Omega_1 \setminus \Gamma_1$ and $\Omega_2 \setminus \Gamma_2$ and as Ω'_1 the locally compact space obtained from Ω by completing first $\vartheta_1(\mathbf{S}_{n-1})$ to $\vartheta_1(\mathbb{B}_n)$ and deleting then ω . By the Product Theorem (Proposition 2.3.1 a)),

$$K_i(\mathcal{C}_0(\Omega'_3, F)) \approx K_i(\mathcal{C}_0(\Omega_1 \setminus \Gamma_1, F)) \times K_i(\mathcal{C}_0(\Omega_2 \setminus \Gamma_2, F)).$$

Since $\Omega'_2 \setminus \Omega'_3$ is homeomorphic to $\mathbf{S}_{n-1} \setminus \{\omega\}$, we get by Theorem 3.2.2 e_1),

$$K_i(\mathcal{C}_0(\Omega'_2 \setminus \Omega'_3, F)) \approx K_{i+n-1}(F).$$

Thus by the topological triple (Proposition 2.1.11 b_3)) (and Theorem 3.1.2 b)),

$$\begin{aligned} K_i(\mathcal{C}_0(\Omega \setminus \{\vartheta_1(\omega)\}, F)) &\approx K_i(\mathcal{C}_0(\Omega'_2, F)) \approx \\ &\approx K_i(\mathcal{C}_0(\Omega'_3, F)) \times K_i(\mathcal{C}_0(\Omega'_2 \setminus \Omega'_3, F)) \approx \\ &\approx K_i(\mathcal{C}_0(\Omega_1 \setminus \Gamma_1, F)) \times K_i(\mathcal{C}_0(\Omega_2 \setminus \Gamma_2, F)) \times K_{i+n-1}(F). \end{aligned} \quad \blacksquare$$

COROLLARY 3.5.15 *If S_g is an orientable compact connected surface of genus $g \in \mathbb{N}$ and Γ is a nonempty finite subset of S_g then*

$$K_i(\mathcal{C}(S_g, F)) \approx K_i(F)^{g+1} \times K_{i+1}(F)^{3g-1},$$

$$K_i(\mathcal{C}(S_g \setminus \Gamma, F)) \approx K_i(F)^g \times K_{i+1}(F)^{3g-2+Card\Gamma}.$$

Assume first Γ is a one-point set $\{\omega\}$. We prove the second assertion in this case by induction with respect to $g \in \mathbb{N}$. By Proposition 3.2.15 b), the assertion holds for $g = 1$. Assume now the assertion holds for $g \in \mathbb{N}$. Let Δ_1 be a closed disc of S_1 , Δ_g a closed disc of S_g , $\omega \in \Delta_1$, and $\omega \in \Delta_g$. $S_{g+1} \setminus \{\omega\}$ can be obtained from the topological sum of $S_1 \setminus \Delta_1$, $S_g \setminus \Delta_g$, and $\mathbf{S}_1 \setminus \{\omega\}$ by pasting $\mathbf{S}_1 \setminus \{\omega\}$ in the boundaries of $\Delta_1 \setminus \{\omega\}$ and $\Delta_g \setminus \{\omega\}$. By the induction hypothesis, since $S_g \setminus \Delta_g$ is homeomorphic to $S_g \setminus \{\omega\}$,

$$K_i(\mathcal{C}_0(S_g \setminus \Delta_g, F)) \approx K_i(F)^g \times K_{i+1}(F)^{3g-1}.$$

By Proposition 3.5.14,

$$K_i(\mathcal{C}_0(S_{g+1} \setminus \{\omega\}, F)) \approx K_i(F)^{g+1} \times K_{i+1}(F)^{3g+2},$$

which finishes the inductive proof.

The first assertion follows now from Alexandroff's K-theorem (Proposition 2.2.1 a)) and the second one from Proposition 2.4.11. ■

The following Example shows a way to generalize Corollary 3.5.15.

EXAMPLE 3.5.16 *Let Ω be the compact space obtained from the topological sum of $\mathbf{S}_1 \times \mathbf{S}_2 \setminus \Delta$, $\mathbf{S}_1 \times \mathbf{S}_1 \times \mathbf{S}_1 \setminus \Delta'$, and \mathbf{S}_2 , where Δ and Δ' denote balls homeomorphic to \mathbb{B}_3 by pasting \mathbf{S}_2 in the boundaries of Δ and Δ' . Then for every nonempty finite subset Γ of Ω ,*

$$K_i(\mathcal{C}(\Omega, F)) \approx K_i(F)^5 \times K_{i+1}(F)^6,$$

$$K_i(\mathcal{C}_0(\Omega \setminus \Gamma, F)) \approx K_i(F)^4 \times K_{i+1}(F)^{5+Card\Gamma}.$$
■

Remark. Let

$$0 \longrightarrow F_1 \xrightarrow{\varphi} F_2 \xrightarrow{\psi} F_3 \longrightarrow 0,$$

$$0 \longrightarrow G_1 \xrightarrow{\varphi'} G_2 \xrightarrow{\psi'} G_3 \longrightarrow 0$$

be exact sequences in \mathfrak{M}_E and $\lambda : F_3 \longrightarrow G_3$ and isomorphism in \mathfrak{M}_E . Then

$$H := \{ (x, y) \in F_2 \times G_2 \mid \psi' y = \lambda \psi x \}$$

is a C^* -subalgebra of $F_2 \times G_2$ containing the ideal $F_1 \times G_1$ of $F_2 \times G_2$. H corresponds to the operation of pasting F_2 and G_2 in \mathfrak{M}_E .

Chapter 4

Some Supplementary Results

Throughout this chapter F denotes an E - C^* -algebra.

4.1 Full E - C^* -algebras

DEFINITION 4.1.1 A **full E - C^* -algebra** is a unital C^* -algebra F for which E is a canonical unital C^* -subalgebra such that $\alpha x = x\alpha$ for all $(\alpha, x) \in E \times F$. Every full E - C^* -algebra is canonically an E - C^* -algebra, the exterior multiplication being the restriction of the interior multiplication. We denote by \mathfrak{C}_E the category of full E - C^* -algebras for which the morphisms are the unital E -linear C^* -homomorphisms. In particular $\mathfrak{C}_{\mathbb{C}}$ is the category of all unital C^* -algebras with unital C^* -homomorphisms. A **full E - C^* -subalgebra of F** is a C^* -subalgebra of F containing E . An isomorphism of full E - C^* -algebras is also called **E - C^* -isomorphism**.

If $\prod_{j \in J} F_j$ is a finite family of full E - C^* -algebras, $J \neq \emptyset$, then $\prod_{j \in J} F_j$ is a full E - C^* -algebra, the canonical embedding $E \rightarrow \prod_{j \in J} F_j$ being given by

$$E \longrightarrow \prod_{j \in J} F_j, \quad \alpha \longmapsto (\alpha)_{j \in J}.$$

If F is a full E - C^* -algebra and G a unital C^* -algebra then the map

$$E \longrightarrow F \otimes G, \quad \alpha \longmapsto \alpha \otimes 1_G$$

is an injective C^* -homomorphism. In particular, the E - C^* -algebra $F \otimes G$ has a canonical structure of a full E - C^* -algebra.

PROPOSITION 4.1.2 Let F be an E - C^* -algebra. We denote by \check{F} the vector space $E \times F$ endowed with the bilinear map

$$(E \times F) \times (E \times F) \longrightarrow E \times F, \quad ((\alpha, x), (\beta, y)) \longmapsto (\alpha\beta, \alpha y + \beta x + xy)$$

and with the involution

$$E \times F \longrightarrow E \times F, \quad (\alpha, x) \longmapsto (\alpha^*, x^*).$$

- a) \check{F} is an involutive unital algebra with $(1_E, 0)$ as unit and $\{(\alpha, 0) \mid \alpha \in E\}$ is a unital involutive subalgebra of \check{F} isomorphic to E .

b) If E and F are C^* -subalgebras of a C^* -algebra G then the map

$$\varphi : \check{F} \longrightarrow E \times G, \quad (\alpha, x) \longmapsto (\alpha, \alpha + x)$$

is an injective involutive algebra homomorphism with closed image

$$\{ (\alpha, y) \in E \times G \mid \alpha - y \in F \} .$$

In particular $\varphi(\check{F})$ is a C^* -subalgebra of $E \times G$ and there is a norm on \check{F} with respect to which \check{F} is a C^* -algebra.

c) There is a unique C^* -norm on \check{F} making it a C^* -algebra. Moreover \check{F} is a full E - C^* -algebra and F may be identified with the closed ideal

$$\{ (0, x) \mid x \in F \}$$

of \check{F} . We shall always consider \check{F} endowed with the structure of a full E - C^* -algebra.

d) If F is a full E - C^* -algebra then the map

$$\check{F} \longrightarrow E \times F, \quad (\alpha, x) \longmapsto (\alpha, \alpha + x)$$

is an isomorphism of E - C^* -algebras with inverse

$$E \times F \longrightarrow \check{F}, \quad (\alpha, x) \longmapsto (\alpha, x - \alpha) .$$

e) If $E = \mathbb{C}$ then \check{F} is the unitization \tilde{F} of F .

a) is easy to verify.

b) Only the assertion that the image of φ is closed needs a proof. Let $(\alpha, x) \in \overline{\varphi(\check{F})}$. There are sequences $(\alpha_n)_{n \in \mathbb{N}}$ and $(x_n)_{n \in \mathbb{N}}$ in E and F , respectively, such that

$$\lim_{n \rightarrow \infty} (\alpha_n, \alpha_n + x_n) = (\alpha, x) .$$

It follows

$$\alpha = \lim_{n \rightarrow \infty} \alpha_n \in E, \quad x - \alpha = \lim_{n \rightarrow \infty} x_n \in F, \quad (\alpha, x) = \varphi(\alpha, x - \alpha) \in \varphi(\check{F}) .$$

Thus $\varphi(\check{F})$ is closed.

c) Let Ω be the spectrum of E and \tilde{F} the unitization of F . Then E and F are C^* -subalgebras of the C^* -algebra $\mathcal{C}(\Omega, \tilde{F})$ and the assertion follows from b).

d) follows from c) and b).

e) is obvious. ■

EXAMPLE 4.1.3 *Let F be a commutative E - C^* -algebra.*

a) \check{F} is commutative. We denote by Ω_E , Ω_F , and $\Omega_{\check{F}}$ the spectra of E , F , and \check{F} , respectively.

b) Ω_F is homeomorphic to an open set Ω' of $\Omega_{\check{F}}$ such that $F \approx \mathcal{C}_0(\Omega', \mathbb{C})$.

c) There is a unique surjective continuous map $\vartheta : \Omega_{\check{F}} \longrightarrow \Omega_E$ such that if we put

$$\phi : E \approx \mathcal{C}(\Omega_E, \mathbb{C}) \longrightarrow \check{F} \approx \mathcal{C}(\Omega_{\check{F}}, \mathbb{C}), \quad \alpha \longmapsto \alpha \circ \vartheta$$

then ϕ is an injective continuous C^* -homomorphism (so we may identify E with $\phi(E)$).

d) The restriction of ϑ to $\Omega_{\check{F}} \setminus \Omega'$ is a homeomorphism.

e) If F is unital then $\Omega_{\check{F}}$ is homeomorphic to the topological sum of Ω_E and Ω_F .

a) is easy to see.

b) follows from the fact that F may be identified with a closed ideal of \check{F} (Proposition 4.1.2 c)).

c) is proved in [1] Proposition 4.1.2.15.

d) Let $\omega \in \Omega_E$ and put

$$\omega' : \check{F} \longrightarrow \mathbb{C}, \quad (\alpha, x) \longmapsto \alpha(\omega).$$

Then $\omega' \in \Omega_{\check{F}} \setminus \Omega'$ and $\vartheta(\omega') = \omega$, so $\vartheta|(\Omega_{\check{F}} \setminus \Omega')$ is surjective.

Let $\omega_1, \omega_2 \in \Omega_{\check{F}} \setminus \Omega'$, $\omega_1 \neq \omega_2$. There is an $(\alpha, x) \in \check{F}$ with

$$\langle (\alpha, x), \omega_1 \rangle \neq \langle (\alpha, x), \omega_2 \rangle.$$

Since $\langle (\alpha, x), \omega_j \rangle = \langle \alpha, \omega_j \rangle$ for every $j \in \{1, 2\}$, $\vartheta|(\Omega_{\check{F}} \setminus \Omega')$ is injective.

e) follows from d) since in this case Ω' is clopen. ■

Remark. The above d) may be seen as a kind of generalization of Alexandroff's compactification.

DEFINITION 4.1.4 *We put for every E - C^* -algebra F*

$$\begin{aligned}\iota^F : F &\longrightarrow \check{F}, & x &\longmapsto (0, x), \\ \pi^F : \check{F} &\longrightarrow E, & (\alpha, x) &\longmapsto \alpha, \\ \lambda^F : E &\longrightarrow \check{F}, & \alpha &\longmapsto (\alpha, 0), \\ \sigma^F &:= \lambda^F \circ \pi^F.\end{aligned}$$

If $E = \mathbb{C}$ then

$$\check{F} = \tilde{F}, \quad \iota_F = \iota^F, \quad \pi_F = \pi^F, \quad \lambda_F = \lambda^F.$$

All these maps are E -linear C^* -homomorphisms,

$$\pi^F \circ \iota^F = 0, \quad \pi^F \circ \lambda^F = id_E, \quad \pi^F \circ \sigma^F = \pi^F,$$

ι^F and λ^F are injective, π^F , λ^F , and σ^F are unital, and

$$0 \longrightarrow F \xrightarrow{\iota^F} \check{F} \begin{matrix} \xrightarrow{\pi^F} \\ \xleftarrow{\lambda^F} \end{matrix} E \longrightarrow 0$$

is a split exact sequence in \mathfrak{M}_E .

PROPOSITION 4.1.5

a) *If $F \xrightarrow{\varphi} F'$ is a morphism in \mathfrak{M}_E then the map*

$$\check{\varphi} : \check{F} \longrightarrow \check{F}', \quad (\alpha, x) \longmapsto (\alpha, \varphi x)$$

is an involutive unital algebra homomorphism, injective or surjective if φ is so. If $F = F'$ and if φ is the identity map then $\check{\varphi}$ is also the identity map.

b) Let F_1, F_2, F_3 be E - C^* -algebras and let $\varphi : F_1 \rightarrow F_2$ and $\psi : F_2 \rightarrow F_3$ be E -linear C^* -homomorphisms. Then $\widetilde{\psi \circ \varphi} = \widetilde{\psi} \circ \widetilde{\varphi}$. ■

Remark. If $E = \mathbb{C}$ then $\check{\varphi} = \tilde{\varphi}$.

EXAMPLE 4.1.6 Let F be a full E - C^* -algebra and F' a closed ideal of F .

a) F' endowed with the exterior multiplication

$$E \times F' \longrightarrow F', \quad (\alpha, x) \longmapsto \alpha x$$

is an E - C^* -algebra.

b) The map

$$\check{F}' \longrightarrow E \times F, \quad (\alpha, x) \longmapsto (\alpha, \alpha + x)$$

is an injective E -linear C^* -homomorphism with image

$$\{ (\alpha, x) \in E \times F \mid \alpha - x \in F' \}.$$

c) \mathfrak{C}_E is a full subcategory of \mathfrak{M}_E . ■

PROPOSITION 4.1.7 Let F be a full E - C^* -algebra and J a finite set.

a) $F^J = F \otimes l^2(J)$ endowed with the maps

$$F \times F^J \longrightarrow F^J, \quad (x, \xi) \longmapsto (x\xi_j)_{j \in J},$$

$$F^J \times F \longrightarrow F^J, \quad (\xi, x) \longmapsto (\xi_j x)_{j \in J},$$

$$F^J \times F^J \longrightarrow F, \quad (\xi, \eta) \longmapsto \sum_{j \in J} \eta_j^* \xi_j$$

is a unital Hilbert F -module ([1] Proposition 5.6.4.2 c)).

b) Let $\mathcal{L}(F^J)$ be the Banach space of operators on F^J . The set $\mathcal{L}_F(F^J)$ of adjointable operators on F^J is a Banach subspace of $\mathcal{L}(F^J)$. $\mathcal{L}_F(F^J)$ endowed with the restriction of the norm of $\mathcal{L}(F^J)$ it is a full E - C^* -algebra ([1] Theorem 5.6.1.11 d), [1] Proposition 5.6.1.8 g), h)). ■

PROPOSITION 4.1.8 *For every E - C^* -algebra F the sequence*

$$0 \longrightarrow K_i(F) \xrightarrow{K_i(\iota^F)} K_i(\check{F}) \xrightleftharpoons[K_i(\lambda^F)]{K_i(\pi^F)} K_i(E) \longrightarrow 0$$

is split exact and the map

$$K_i(F) \times K_i(E) \longrightarrow K_i(\check{F}), \quad (a, b) \longmapsto K_i(\iota^F)a + K_i(\lambda^F)b$$

is a group isomorphism.

Since the sequence in \mathfrak{M}_E

$$0 \longrightarrow F \xrightarrow{\iota^F} \check{F} \xrightleftharpoons[\lambda^F]{\pi^F} E \longrightarrow 0$$

is split exact the assertion follows from the split exact axiom (Axiom 1.2.3). ■

COROLLARY 4.1.9 *Let G be a C^* -algebra.*

a) *The sequence in \mathfrak{M}_E*

$$0 \longrightarrow F \otimes G \xrightarrow{\iota^F \otimes id_G} \check{F} \otimes G \xrightleftharpoons[\lambda^F \otimes id_G]{\pi^F \otimes id_G} E \otimes G \longrightarrow 0$$

is split exact.

b) *The sequence*

$$0 \longrightarrow K_i(F \otimes G) \xrightarrow{K_i(\iota^F \otimes id_G)} K_i(\check{F} \otimes G) \xrightleftharpoons[K_i(\lambda^F \otimes id_G)]{K_i(\pi^F \otimes id_G)} K_i(E \otimes G) \longrightarrow 0$$

is split exact and the map

$$K_i(E \otimes G) \times K_i(F \otimes G) \longrightarrow K_i(\check{F} \otimes G),$$

$$(a, b) \longmapsto K_i(\lambda^F \otimes id_G)a + K_i(\iota^F \otimes id_G)b$$

is a group isomorphism.

c) *Let $F \xrightarrow{\Phi} F'$ be a morphism in \mathfrak{M}_E and $G \xrightarrow{\Psi} G'$ a morphism in \mathfrak{M}_E . If we identify the isomorphic groups of b) then*

$$K_i(\check{\Phi} \otimes \Psi) : K_i(\check{F} \otimes G) \longrightarrow K_i(\check{F}' \otimes G'),$$

$$(a, b) \longmapsto (K_i(id_E \otimes \Psi)a, K_i(\Phi \otimes \Psi)b)$$

is a group isomorphism.

a) follows from Proposition 1.4.8 a).

b) follows from a) and the split exact axiom (Axiom 1.2.3).

c) follows from b) and the commutativity of the following diagram:

$$\begin{array}{ccccc}
 F \otimes G & \xrightarrow{\iota^F \otimes id_G} & \check{F} \otimes G & \xleftarrow{\lambda^F \otimes id_G} & E \otimes G \\
 \varphi \otimes \psi \downarrow & & \check{\varphi} \otimes \psi \downarrow & & \downarrow id_E \otimes \psi \\
 F' \otimes G' & \xrightarrow{\iota^{F'} \otimes id_{G'}} & \check{F}' \otimes G' & \xleftarrow{\lambda^{F'} \otimes id_{G'}} & E \otimes G'
 \end{array} .$$

COROLLARY 4.1.10 *Let $F \xrightarrow{\check{\varphi}_1} F'$ and $F \xrightarrow{\check{\varphi}_2} F'$ be morphisms in \mathfrak{M}_E . If F is K -null then $K_i(\check{\varphi}_1) = K_i(\check{\varphi}_2)$.*

By Proposition 4.1.8, the map

$$K_i(F) \times K_i(E) \longrightarrow K_i(\check{F}), \quad (a, b) \longmapsto K_i(\iota^F) a + K_i(\lambda^F) b$$

is a group isomorphism. Since F is K -null, $K_i(\lambda^F)$ is a group isomorphism. We get from $\check{\varphi}_1 \circ \lambda^F = \check{\varphi}_2 \circ \lambda^F$,

$$K_i(\check{\varphi}_1) \circ K_i(\lambda^F) = K_i(\check{\varphi}_2) \circ K_i(\lambda^F), \quad K_i(\check{\varphi}_1) = K_i(\check{\varphi}_2) .$$

4.2 Continuity and Stability

AXIOM 4.2.1 (Continuity axiom) *If $\{(F_j)_{j \in J}, (\varphi_{j,k})_{j,k \in I}\}$ is an inductive system in \mathfrak{M}_E such that $\varphi_{j,k}$ are injective for all $j, k \in J, k < j$, and if $\{F, (\varphi_j)_{j \in J}\}$ denotes its inductive limit in \mathfrak{M}_E then $\{K_i(F), (K_i(\varphi_j))_{j \in J}\}$ is the inductive limit of the inductive system $\{(K_i(F_j))_{j \in J}, (K_i(\varphi_{j,k}))_{j,k \in J}\}$.*

PROPOSITION 4.2.2 *If Ω is a totally disconnected compact space then*

$$K_i(\mathcal{C}(\Omega, F)) \approx \left\{ a \in K_i(F)^\Omega \mid a(\Omega) \text{ is finite} \right\} .$$

Let Ξ be the set of clopen partitions of Ω ordered by fineness and for every $\Theta := (\Omega_j)_{j \in J} \in \Xi$ and $x \in F^\Theta$ put

$$\tilde{x} : \Omega \longrightarrow F, \quad \omega \longmapsto x(j) \quad \text{for } \omega \in \Omega_j.$$

Then the map

$$F^\Theta \longrightarrow \mathcal{C}(\Omega, F), \quad x \longmapsto \tilde{x}$$

is an injective E - C^* -homomorphism for every $\Theta \in \Xi$ and $\mathcal{C}(\Omega, F)$ is isomorphic to the corresponding inductive limit in \mathfrak{M}_E of $(F^\Theta)_{\Theta \in \Xi}$. By Lemma 2.1.4 c), $K_i(F^\Theta) \approx K_i(F)^\Theta$ for every $\Theta \in \Xi$ and the assertion follows from the continuity axiom (Axiom 4.2.1). ■

PROPOSITION 4.2.3 *Let ξ be an ordinal number, $(\Omega_\eta)_{\eta < \xi}$ a family of path connected, non-compact, locally compact spaces, and $\omega_\eta \in \Omega_\eta$ for every $\eta < \xi$. We denote by Ω^ξ the locally compact space obtained by endowing the disjoint union of the family of sets $(\Omega_\eta)_{\eta < \xi}$ with the topology for which a subset U of Ω^ξ is open if it has the following properties:*

- 1) $\Omega_\eta \cap U$ is open for every $\eta < \xi$.
- 2) If $\omega_\eta \in U$ for some $\eta < \xi$ and if there is a $\zeta < \eta$ with $\eta = \zeta + 1$ then $\Omega_\zeta \setminus U$ is compact.
- 3) If $\omega_\eta \in U$ for some limit ordinal number $\eta < \xi$ then there is a $\zeta < \eta$ such that $\bigcup_{\zeta < \zeta' < \eta} \Omega_{\zeta'} \subset U$.

If $K_i(\mathcal{C}_0(\Omega_\eta, F)) = 0$ for all $\eta < \xi$ then $K_i\left(\mathcal{C}_0\left(\Omega^\xi, F\right)\right) = 0$.

The assertion is trivial for $\xi = 0$. We prove the general case by transfinite induction. If $\xi = \eta + 1$ for some $\eta < \xi$ for which the assertion holds then by Corollary 3.5.4, the assertion holds also for ξ . If ξ is a limit ordinal number and the assertion holds for every $\eta < \xi$ then by the continuity axiom (Axiom 4.2.1) the assertion holds also for ξ since $\mathcal{C}_0(\Omega^\xi, F)$ is the inductive limit of the inductive system $\{\mathcal{C}_0(\Omega_\eta, F) \mid \eta < \xi\}$. ■

Remark. If $\Omega_\eta = [0, 1[$ for every $\eta < \xi$ then Ω^ξ is "one-dimensional".

LEMMA 4.2.4 *Let $\{(F_j)_{j \in J}, (\varphi_{j,k})_{j,k \in J}\}$ be an inductive system in \mathfrak{M}_E , $\{F, (\varphi_j)_{j \in J}\}$ its inductive limit in \mathfrak{M}_E , G an E - C^* -algebra, and for every $j \in J$ an injective morphism $\psi_j : F_j \rightarrow G$ in \mathfrak{M}_E such that $\psi_j = \psi_k \circ \varphi_{k,j}$ for all $j, k \in J$, $j < k$. Then the morphism $\psi : F \rightarrow G$ in \mathfrak{M}_E such that $\psi_j = \psi \circ \varphi_j$ for all $j, k \in J$, $j < k$, ([5] Theorem L.2.1) is injective.*

For $j \in J$ and $x \in F_j$,

$$\|\varphi_j x\| \leq \|x\| = \|\psi_j x\| = \|\psi \varphi_j x\| \leq \|\varphi_j x\|,$$

so ψ preserves the norms on $\varphi_j(F_j)$. Since $\bigcup_{j \in J} \varphi_j(F_j)$ is dense in F , ψ preserves the norms, i.e. it is injective. ■

PROPOSITION 4.2.5 *Let $\{(G_j)_{j \in J}, (\varphi_{j,k})_{j,k \in J}\}$ be an inductive system in $\mathfrak{M}_{\mathbb{C}}$ such that $\varphi_{j,k}$ are injective for all $j, k \in J$, $k < j$, and let $\{G, (\varphi_j)_{j \in J}\}$ be its inductive limit in $\mathfrak{M}_{\mathbb{C}}$. If $\{F', (\psi_j)_{j \in J}\}$ denotes the inductive limit in \mathfrak{M}_E of the inductive system $\{(F \otimes G_j)_{j \in J}, (id_F \otimes \varphi_{j,k})_{j,k \in J}\}$ in \mathfrak{M}_E and $\psi : F' \rightarrow F \otimes G$ denotes the morphism in \mathfrak{M}_E such that $\psi \circ \psi_j = id_F \otimes \varphi_j$ for all $j \in J$ ([5] Theorem L.2.1) then ψ is an isomorphism.*

By [5] Corollary T.5.19, $id_F \otimes \varphi_j$ are injective for all $j \in J$. By Lemma 4.2.4, ψ is injective. Since

$$F \otimes \left(\bigcup_{j \in J} G_j \right) \subset Im \psi,$$

ψ is surjective and so it is an isomorphism. ■

COROLLARY 4.2.6 *If $\{(G_j)_{j \in J}, (\varphi_{j,k})_{j,k \in J}\}$ is an inductive system in $\mathfrak{M}_{\mathbb{C}}$ such that $\varphi_{j,k}$ are injective for all $j, k \in J$, $k < j$, and if $\{G, (\varphi_j)_{j \in J}\}$ is its inductive limit in $\mathfrak{M}_{\mathbb{C}}$ then $\{K_i(F \otimes G), (K_i(id_F \otimes \varphi_j))_{j \in J}\}$ is the inductive limit of the inductive system $\{(K_i(F \otimes G_j))_{j \in J}, (K_i(id_F \otimes \varphi_{j,k}))_{j,k \in J}\}$. In particular if G_j is Υ -null for every $j \in J$ then G is also Υ -null.*

By [5] Corollary T.5.19, $id_F \otimes \varphi_{j,k}$ are injective for all $j, k \in J$, $k < j$. By Proposition 4.2.5, $\{F \otimes G, (id_F \otimes \varphi_j)_{j \in J}\}$ may be identified with the inductive limit in \mathfrak{M}_E of the inductive system $\{(F \otimes G_j)_{j \in J}, (id_F \otimes \varphi_{j,k})_{j,k \in J}\}$ in \mathfrak{M}_E and the assertion follows from the continuity axiom (Axiom 4.2.1). ■

COROLLARY 4.2.7 Let $(G_j)_{j \in J}$ be an infinite family in Υ_1 , \mathfrak{J} the set of nonempty finite subsets of J ordered by inclusion, and for all $K, L \in \mathfrak{J}$, $K \subset L$, put $G_K := \bigotimes_{j \in K} G_j$ and

$$\varphi(L, K) : G_K \longrightarrow G_L, \quad \bigotimes_{j \in K} x_j \longmapsto \bigotimes_{j \in L} y_j,$$

where

$$y_j := \begin{cases} x_j & \text{if } j \in K \\ 1_{G_j} & \text{if } j \in L \setminus K \end{cases}.$$

Then $\{(G_K)_{K \in \mathfrak{J}}, (\varphi(L, K))_{K, L \in \mathfrak{J}}\}$ is an inductive system in $\mathfrak{M}_{\mathbb{C}}$ and its limit belongs to Υ_1 .

We denote by $\{G, (\varphi(K))_{K \in \mathfrak{J}}\}$ the above inductive limit. By Proposition 1.6.5, $G_K \in \Upsilon_1$ for all $K \in \mathfrak{J}$ so by Corollary 4.2.6, $p(G) = 1$, $q(G) = 0$. Let $F \xrightarrow{\phi} F'$ be a morphism in \mathfrak{M}_E and let $K \in \mathfrak{J}$. Then the diagram

$$\begin{array}{ccccc} F & \xrightarrow{\phi_{G_K, F}} & F \otimes G_K & \xrightarrow{id_F \otimes \varphi(K)} & F \otimes G \\ \phi \downarrow & & \downarrow \phi \otimes id_{G_K} & & \downarrow \phi \otimes id_G \\ F' & \xrightarrow{\phi_{G_K, F'}} & F' \otimes G_K & \xrightarrow{id_{F'} \otimes \varphi(K)} & F' \otimes G \end{array}$$

is commutative. Since

$$\phi_{G, F} = (id_F \otimes \varphi(K)) \circ \phi_{G_K, F}, \quad \phi_{G, F'} = (id_{F'} \otimes \varphi(K)) \circ \phi_{G_K, F'},$$

the diagrams

$$\begin{array}{ccc} F & \xrightarrow{\phi_{G, F}} & F \otimes G \\ \phi \downarrow & & \downarrow \phi \otimes id_G \\ F' & \xrightarrow{\phi_{G, F'}} & F' \otimes G \end{array} \quad \begin{array}{ccc} K_i(F) & \xrightarrow{K_i(\phi_{G, F})} & K_i(F \otimes G) \\ K_i(\phi) \downarrow & & \downarrow K_i(\phi \otimes id_G) \\ K_i(F') & \xrightarrow{K_i(\phi_{G, F'})} & K_i(F' \otimes G) \end{array}$$

are commutative and so $G \in \Upsilon_1$. ■

COROLLARY 4.2.8 Let $\{(G_j)_{j \in J}, (\varphi_{j,k})_{j,k \in J}\}$ be an inductive system in $\mathfrak{M}_{\mathbb{C}}$ such that $\varphi_{k,j}$ are injective for all $j, k \in J$, $j < k$, and let $\{G, (\varphi_j)_{j \in J}\}$ be its inductive limit. We

assume that for all $j, k \in J$, $j < k$,

$$G_j, G_k \in \Upsilon, \quad \Phi_{i, G_k, F} = K_i(id_F \otimes \varphi_{k,j}) \circ \Phi_{i, G_j, F}.$$

Then

$$G \in \Upsilon, \quad \Phi_{i, G, F} = K_i(id_F \otimes \varphi_j) \circ \Phi_{i, G_j, F}$$

for all $j \in J$.

By Corollary 4.2.6, $\{K_i(F \otimes G), (K_i(id_F \otimes \varphi_j))_{j \in J}\}$ is the inductive limit of the inductive system $\{(K_i(F \otimes G_j))_{j \in J}, (K_i(id_F \otimes \varphi_{j,k}))_{j,k \in J}\}$. By the hypothesis of the Corollary,

$$K_i(id_F \otimes \varphi_{k,j}) : K_i(F \otimes G_j) \longrightarrow K_i(F \otimes G_k)$$

is a group isomorphism for all $j, k \in J$, $j < k$, so

$$K_i(id_F \otimes \varphi_j) : K_i(F \otimes G_j) \longrightarrow K_i(F \otimes G)$$

is also a group isomorphism for all $j \in J$. Let $F \xrightarrow{\phi} F'$ be a morphism in \mathfrak{M}_E . The assertion follows from the commutativity of the diagram

$$\begin{array}{ccc} K_i(F)^{p(G_j)} \times K_{i+1}(F)^{q(G_j)} & \xrightarrow{K_i(\phi)^{p(G_j)} \times K_{i+1}(\phi)^{q(G_j)}} & A \\ \Phi_{i, G_j, F} \downarrow & & \Phi_{i, G_j, F'} \downarrow \\ K_i(F \otimes G_j) & \xrightarrow{K_i(\phi \otimes G_j)} & K_i(F' \otimes G_j) \\ K_i(id_F \otimes \varphi_j) \downarrow & & K_i(id_{F'} \otimes \varphi_j) \downarrow \\ K_i(F \otimes G) & \xrightarrow{K_i(\phi \otimes id_G)} & K_i(F' \otimes G) \end{array}$$

where $A := K_i(F')^{p(G_j)} \times K_{i+1}(F')^{q(G_j)}$. ■

DEFINITION 4.2.9 We denote for every family $(\mathcal{G}_j)_{j \in J}$ of additive groups by $\sum_{j \in J} \mathcal{G}_j$ its direct sum i.e.

$$\sum_{j \in J} \mathcal{G}_j := \left\{ a \in \prod_{j \in J} \mathcal{G}_j \mid \{ j \in J \mid a_j \neq 0 \} \text{ is finite} \right\}.$$

PROPOSITION 4.2.10 *If $(F_j)_{j \in J}$ is a family of E - C^* -algebras and F is its C^* -direct sum ([1] Example 4.1.1.6) then*

$$K_i(F) \approx \sum_{j \in J} K_i(F_j) .$$

In particular, the C^ -direct sum of a family of K -null E - C^* -algebras is K -null.*

If J is finite then the assertion follows from Proposition 1.3.3. The general case follows now from the continuity (Axiom 4.2.1). ■

COROLLARY 4.2.11 *If $(\Omega_j)_{j \in J}$ is a family of locally compact spaces and Ω is its topological sum then*

$$\begin{aligned} K_i(\mathcal{C}_0(\Omega, F)) &\approx \sum_{j \in J} K_i(\mathcal{C}_0(\Omega_j, F)) \approx \\ &\approx \left\{ a \in \prod_{j \in J} K_i(\mathcal{C}_0(\Omega_j, F)) \mid \{ j \in J \mid a_j \neq 0 \} \text{ is finite} \right\} . \end{aligned}$$

By Proposition 4.2.5, $\mathcal{C}_0(\Omega, F)$ is the direct sum of the family $(\mathcal{C}_0(\Omega_j, F))_{j \in J}$ and the assertion follows from Proposition 4.2.10. ■

PROPOSITION 4.2.12 *If ξ is an ordinal number endowed with its usual topology then*

$$K_i(\mathcal{C}_0(\xi, F)) \approx \sum_{\eta < \xi} K_i(F).$$

We prove the assertion by transfinite induction. If ξ is not a limit ordinal number then the assertion follows from Corollary 2.3.4 a). Assume ξ is a limit ordinal number and for all $\eta < \xi < \xi$ let $\varphi_{\zeta, \eta} : \mathcal{C}_0(\eta, F) \longrightarrow \mathcal{C}_0(\zeta, F)$ be the inclusion map. By Proposition 4.2.5, $\mathcal{C}_0(\xi, F)$ may be identified with the inductive limit in \mathfrak{M}_E of the inductive system $\{(\mathcal{C}_0(\eta, F))_{\eta < \xi}, (\varphi_{\zeta, \eta})_{\eta < \zeta < \xi}\}$ in \mathfrak{M}_E . Thus the assertion follows from the continuity axiom (Axiom 4.2.1) and the induction hypothesis. ■

DEFINITION 4.2.13 *We denote for every $n \in \mathbb{N}$ by $M(n)$ the C^* -algebra of $n \times n$ -matrices with entries in \mathbb{C} .*

AXIOM 4.2.14 (Stability axiom) *There is an $h \in \mathbb{N}$, $h \neq 1$, such that*

$$M(h) \in \Upsilon, \quad p(M(h)) = 1, \quad q(M(h)) = 0,$$

$$\Phi_{i,M(h),F} = K_i(id_F \otimes \varphi) \circ \Phi_{i,\mathbb{C},F},$$

where

$$\varphi : \mathbb{C} \longrightarrow M(h), \quad \alpha \longmapsto \begin{pmatrix} \alpha & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

PROPOSITION 4.2.15 *We put for all $j, k \in \mathbb{N}^*$, $j < k$,*

$$\varphi_{k,j} : M(h^j) \longrightarrow M(h^k), \quad x \longmapsto \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

a) *For all $j \in \mathbb{N}$,*

$$M(h^j) \in \Upsilon, \quad p(M(h^j)) = 1, \quad q(M(h^j)) = 0,$$

$$\Phi_{i,M(h^j),F} = K_i(id_F \otimes \varphi_{j,0}) \circ \Phi_{i,\mathbb{C},F}.$$

b) *For all $j, k \in \mathbb{N}^*$, $j < k$,*

$$\Phi_{i,M(h^k),F} = K_i(id_F \otimes \varphi_{k,j}) \circ \Phi_{i,M(h^j),F}$$

and $K_i(id_F \otimes \varphi_{k,j})$ is a group isomorphism.

a) We prove the assertion by induction with respect to $j \in \mathbb{N}$. For $j = 1$ the assertion is exactly the Stability axiom (Axiom 4.2.14). Let $j > 1$ and assume the assertion holds for $j - 1$. With the notation of Proposition 1.5.4 b),

$$(id_{F \otimes M(h)} \otimes \varphi_{(j-1),0}) \circ \Phi_{\mathbb{C},F \otimes M(h)} \circ (id_F \otimes \varphi_{1,0}) = id_F \otimes \varphi_{j,0},$$

so by the above and by the induction hypothesis,

$$K_i(id_F \otimes \varphi_{j,0}) \circ \Phi_{i,\mathbb{C},F} =$$

$$\begin{aligned}
 &= K_i(id_F \otimes M(h)) \otimes \varphi_{(j-1),0}) \circ \Phi_{i,\mathbf{C},F \otimes M(h)} \circ K_i(id_F \otimes \varphi_{1,0}) \circ \Phi_{i,\mathbf{C},F} = \\
 &= \Phi_{i,M(h^{j-1}),F \otimes M(h)} \circ \Phi_{i,M(h),F} .
 \end{aligned}$$

Thus

$$K_i(id_F \otimes \varphi_{j,0}) \circ \Phi_{i,\mathbf{C},F} : K_i(F) \longrightarrow K_i(F \otimes M(h^j))$$

is a group isomorphism. Let $F \xrightarrow{\phi} F'$ be a morphism in \mathfrak{M}_E . Since the diagram

$$\begin{array}{ccccc}
 K_i(F) & \xrightarrow{\Phi_{i,\mathbf{C},F}} & K_i(F \otimes M(1)) & \xrightarrow{K_i(id_F \otimes \varphi_{j,0})} & K_i(F \otimes M(h^j)) \\
 K_i(\phi) \downarrow & & \downarrow K_i(\phi \otimes id_{M(1)}) & & \downarrow K_i(\phi \otimes id_{M(h^j)}) \\
 K_i(F') & \xrightarrow{\Phi_{i,\mathbf{C},F'}} & K_i(F' \otimes M(1)) & \xrightarrow{K_i(id_{F'} \otimes \varphi_{j,0})} & K_i(F' \otimes M(h^j))
 \end{array}$$

is commutative, we may take

$$\Phi_{i,M(h^j),F} = K_i(id_F \otimes \varphi_{j,0}) \circ \Phi_{i,\mathbf{C},F} .$$

b) By a),

$$\begin{aligned}
 K_i(id_F \otimes \varphi_{k,j}) \circ \Phi_{i,M(h^j),F} &= K_i(id_F \otimes \varphi_{k,j}) \circ K_i(id_F \otimes \varphi_{j,0}) \circ \Phi_{i,\mathbf{C},F} = \\
 &= K_i(id_F \otimes \varphi_{k,0}) = \Phi_{i,M(h^k),F} .
 \end{aligned}$$

■

THEOREM 4.2.16 *Let H be an infinite-dimensional Hilbert space and $\mathcal{K}(H)$ the C^* -algebra of compact operators on H . Then*

$$\mathcal{K}(H) \in \Upsilon, \quad p(\mathcal{K}(H)) = 1, \quad q(\mathcal{K}(H)) = 0,$$

$$\Phi_{i,\mathcal{K}(H),F} = K_i(id_F \otimes \varphi) \circ \Phi_{i,\mathbf{C},F},$$

where $\varphi : \mathbf{C} \longrightarrow \mathcal{K}(H)$ is an inclusion map.

Let Ξ be the set of subspaces of H of dimension h^j for some $j \in \mathbb{N}^*$ ordered by inclusion and for every $K \in \Xi$ let π_K be the orthogonal projection of H on K and $G_K := \pi_K \mathcal{K}(H) \pi_K$. We denote for all $K, L \in \Xi$, $K \subset L$, by

$$\varphi_{L,K} : G_K \longrightarrow G_L, \quad \varphi_K : G_K \longrightarrow \mathcal{K}(H)$$

the inclusion maps. Then $\{(G_K)_{K \in \Xi}, (\varphi_{L,K})_{L,K \in \Xi}\}$ is an inductive system in $\mathfrak{M}_{\mathfrak{C}}$ and $\{\mathcal{K}(H), (\varphi_K)_{K \in \Xi}\}$ is its inductive limit. By Proposition 4.2.15, for $K, L \in \Xi, K \subset L$,

$$G_K, G_L \in \Upsilon, \quad p(G_K) = p(G_L) = 1, \quad q(G_K) = q(G_L) = 0,$$

$$\Phi_{i,G_L,F} = K_i(id_F \otimes \varphi_{L,K}) \circ \Phi_{i,G_K,F},$$

and $K_i(id_F \otimes \varphi_{L,K})$ is a group isomorphism. By Corollary 4.2.8, for $K \in \Xi$,

$$\mathcal{K}(H) \in \Upsilon, \quad \Phi_{i,\mathcal{K}(H),F} = K_i(id_F \otimes \varphi_K) \circ \Phi_{i,G_K,F},$$

so $p(\mathcal{K}(H)) = 1, q(\mathcal{K}(H)) = 0$. ■

