Chapter 1

Theory of Gravitation

In this chapter a theory of gravitation in flat space-time is studied which was considered in several articles by the author.

Let us assume a flat space-time metric. Denote by $\left(x^i\right)$ the co-ordinates of space and time then the line-element can be written

$$(ds)^2 = -\eta_{ij} dx^i dx^j \tag{1.1}$$

Here, $\left(\eta_{ij}\right)$ is a symmetric metric tensor. In addition to the metric tensor a symmetric contra-variant tensor $\left(\eta^{ij}\right)$ is defined by

$$\eta_{ik}\eta^{kj} = \delta_i^{\ j}, \quad \eta^{ik}\eta_{kj} = \delta^i_{\ j}. \tag{1.2}$$

Furthermore, we put

$$\eta = \det\left(\eta_{ij}\right). \tag{1.3}$$

In the special case of a pseudo-Euclidean metric we have

$$\left(x^i\right) = \left(x^1, x^2, x^3, ct\right). \tag{1.4}$$

$\left(x^1, x^2, x^3\right)$ are the Cartesian co-ordinates, t is the time and c is the velocity of light. Then, the metric tensor has the form

$$\left(\eta_{ij}\right) = diag\left(1,1,1,-1\right). \tag{1.5}$$

This is the metric in which the most kinds of fields and matter are described.

1.1 Gravitational Potentials

Similar to Maxwell's theory of Electrodynamics we assume that gravitation is described by a field in space and time. The electro-magnetic field can be described with the aid of a four-vector called the potentials of the field and produced by an electric four-current.

Analogously, the symmetric gravitational potentials $\left(g_{ij}\right)$ are produced by the total energy-momentum of matter and gravitational field. Similar to the equations (1.2) let us define a symmetric tensor $\left(g^{ij}\right)$ by

$$g_{ik}g^{kj} = \delta_i^{\ j}, \quad g^{ik}g_{kj} = \delta^i_{\ j} \tag{1.6}$$

We put

$$G = \det\left(g_{ij}\right). \tag{1.7}$$

In addition to the time t we define the proper-time τ by

$$c^2\left(d\tau\right)^2 = -g_{ij}dx^i dx^j. \tag{1.8}$$

The relation (1.8) is similar to the definition of the line-element (1.1) with the metric tensor $\left(\eta_{ij}\right)$. Therefore, theories of gravitation described by $\left(g_{ij}\right)$ with the proper-time (1.8) and with the line-element (1.1) are called bi-metric theories of gravitation.

1.2 Lagrangian

The theory of gravitation is derived from an invariant Lagrangian which is quadratic in the first order co-variant derivatives of the potentials $\left(g_{ij}\right)$ resp. of the contra-variant tensors $\left(g^{ij}\right)$. The derivatives are relative to the flat space-time metric (1.1) and they are denoted with a bar "/". The Lagrangian has the form

$$L_G = -\left(\frac{-G}{-\eta}\right)^{1/2} g_{kl}g_{mn}g^{ij}\left(g^{km}{}_{/i}g^{\ln}{}_{/j} - \frac{1}{2}g^{kl}{}_{/i}g^{mn}{}_{/j}\right) \tag{1.9}$$

In addition let us introduce the invariant Lagrangian

$$L_\Lambda = -8\Lambda\left(\frac{-G}{-\eta}\right)^{1/2} \tag{1.10}$$

Here, Λ is the cosmological constant. For simplicity we consider dust (no pressure) with the density ρ. The Lagrangian for matter can be written in the form

$$L_M = -\rho g_{ij} u^i u^j \tag{1.11}$$

where (u^i) is the four-velocity. It follows by the use of

$$u^i = \frac{dx^i}{d\tau} \tag{1.12}$$

and relation (1.8)

$$-g_{ij} u^i u^j = c^2 . \tag{1.13}$$

By the introducing of the constant

$$\kappa = \frac{4\pi k}{c^4} \tag{1.14}$$

the whole Lagrangian has the form:

$$L = L_G + L_\Lambda - 8\kappa L_M . \tag{1.15}$$

Here, the constant k denotes the gravitational constant.

1.3 Field Equations

The differential equations for the gravitational potentials (g_{ij}) follow from the variation - equation

$$\partial \int L(-\eta)^{1/2} d^4x . \tag{1.16}$$

From Euler's equations we get by the formulas for the covariant derivatives (see e.g., [Sop 76], p.189 ff)

$$\left[\frac{1}{(-\eta)^{1/2}} \frac{\partial L(-\eta)^{1/2}}{\partial g^{ij}{}_{/k}} \right]_{/k} = \frac{1}{(-\eta)^{1/2}} \frac{\partial L(-\eta)^{1/2}}{\partial g^{ij}} \tag{1.17}$$

implying by the use of (1.15)

$$\left[\frac{\partial L_G}{\partial g^{ij}{}_{/k}}\right]_{/k} = \frac{\partial\left(L_G + L_\Lambda\right)}{\partial g^{ij}} + 8\kappa\frac{\partial L_M}{\partial g^{ij}}\ . \qquad (1.18)$$

We use the following formulas

$$\frac{\partial\left(-G\right)^{1/2}}{\partial g^{ij}} = -\frac{1}{2}\left(-G\right)^{1/2} g_{ij}\ ,\ \ \frac{\partial g_{ij}}{\partial g^{kl}} = -g^{ik} g^{jl}\ .$$

Equation (1.18) implies by the use of these relations and multiplication with g^{lj} the following formula

$$\begin{aligned}&\left[\left(\frac{-G}{-\eta}\right)^{1/2} g^{mn}\left(g_{ik} g^{kj}{}_{/n} - \frac{1}{2}\delta_i^{\ j} g_{kl} g^{kl}{}_{/n}\right)\right]_{m}\\ &= \frac{1}{2}\left(\frac{-G}{-\eta}\right)^{172} g_{mn} g_{kl} g^{jr}\left(g^{mk}{}_{/i} g^{nl}{}_{/r} - \frac{1}{2} g^{mn}{}_{/i} g^{kl}{}_{/r}\right)\\ &+\frac{1}{4}\delta_i^{\ j}\left(L_G + L_\Lambda\right) + 4\kappa\rho g_{im} u^m u^j\end{aligned} \qquad (1.19)$$

These are the field equations of gravitation for dust.

1.4 Equations of Motion and the Energy-Momentum

We will now prove the equivalence of the conservation law of energy-momentum and the equations of motion. It follows from equation (1.18) by multiplication with $g^{kj}{}_{/l}$ and summation

$$\begin{aligned}&\left[g^{mn}{}_{/l}\frac{\partial L_G}{\partial g^{mn}{}_{/k}}\right]_{/k} - \frac{\partial L_G}{\partial g^{mn}{}_{/k}} g^{mn}{}_{/l/k}\\ &= \frac{\partial\left(L_G + L_\Lambda\right)}{\partial g^{mn}} g^{mn}{}_{/l} - 8\kappa\rho g^{mn}{}_{/l} g_{mr} g_{ns} u^r u^s\end{aligned}\ . \qquad (1.20)$$

The mixed energy-momentum tensors of the gravitational field, of vacuum energy (given by the cosmological constant Λ) and of dust are given by

$$T(G)^{j}{}_{i}=\frac{1}{8\kappa}\left[\left(\frac{-G}{-\eta}\right)^{1/2}g_{mn}g_{kl}g^{jr}\left(g^{m,k}{}_{/i}g^{nl}{}_{/r}-\frac{1}{2}g^{mn}{}_{/i}g^{kl}{}_{/r}\right)+\frac{1}{2}\delta^{j}{}_{i}L_{G}\right] \tag{1.21a}$$

$$T(\Lambda)^{j}{}_{i}=\frac{1}{16\kappa}\delta^{j}{}_{i}L_{\Lambda} \tag{1.21b}$$

$$T(M)^{j}{}_{i}=\rho g_{im}u^{m}u^{j} \tag{1.21c}$$

and the corresponding symmetric tensors are defined by

$$T(G)^{ij}=g^{im}T(G)^{j}_{m},\ T(\Lambda)^{ij}=g^{im}T(\Lambda)^{j}_{m},\ T(M)^{ij}=g^{im}T(M)^{j}_{m} \tag{1.22}$$

Put

$$D^{j}{}_{i}=\left[\left(\frac{-G}{-\eta}\right)^{1/2}g^{mn}g_{ik}g^{kj}{}_{/n}\right]_{/m}. \tag{1.23a}$$

Then, the field equations of gravitation (1.19) have the simple form

$$D^{j}{}_{i}-\frac{1}{2}\delta^{j}{}_{i}D^{m}{}_{m}=4\kappa T^{j}{}_{i} \tag{1.23b}$$

Here,

$$T_{i}^{j}=T(G)_{i}^{j}+T(\Lambda)_{i}^{j}+T(M)_{i}^{j}. \tag{1.23c}$$

is the whole energy-momentum tensor of gravitational field, of vacuum energy and of matter.

The equations (1.23) can be rewritten

$$D^{j}{}_{i}=4\kappa\left(T^{j}{}_{i}-\frac{1}{2}\delta^{j}{}_{i}T^{m}{}_{m}\right). \tag{1.24}$$

It is worth to mention that the equations (1.23) are generally co-variant. In particular, the energy-momentum of gravitation is a tensor in contrast to the corresponding pseudo-tensor in Einstein's general relativity.

The field equations of gravitation (1.23b) and (1.24) are formally similar to the corresponding equations of general relativity. Here, $D^{j}{}_{i}$ is a differential operator of order two in divergence form for g^{ij} whereas in general relativity there is instead of that the Ricci tensor. The source of the gravitational field in flat space-time theory of gravitation is the whole energy-momentum tensor inclusive the one of the gravitational field which is not a tensor in Einstein's theory and it does not appear as source for the field.

Relation (1.20) can be rewritten

$$\left[g^{mn}{}_{/k} \frac{\partial L_G}{\partial g^{mn}{}_{/l}} - \delta^l{}_k \left(L_G + L_\Lambda \right) \right]_{/l} = 8\kappa \rho g_{mn/k} u^m u^n ,$$

i.e., we get by the use of (1.21a) and (1.21b)

$$\left(T(G)^m{}_i + T(\Lambda)^m{}_i \right)_{/m} = -\frac{1}{2} \rho g_{mn/i} u^m u^n .$$

This relation becomes by the substitution of (1.21c) and the use of (1.23c)

$$T^m{}_{i/m} = T(M)^m{}_{i/m} - \frac{1}{2} \rho g_{mn/i} u^m u^n = T(M)^m{}_{i/m} - \frac{1}{2} g_{mn/i} T(M)^{mn} .$$

Hence, the conservation of the whole energy-momentum

$$T^m{}_{i/m} = 0 \tag{1.25a}$$

is equivalent with the equations of motion for matter

$$T(M)^m_{i/m} = \frac{1}{2} g_{mn/i} T(M)^{mn} \tag{1.26}$$

The conservation law of the whole energy-momentum (1.25a) can be rewritten

$$\left(\eta^{in} T^m{}_n \right)_{/m} = 0 . \tag{1.25b}$$

The conservation of mass is given by

$$\left(\rho u^m \right)_{/m} = 0 . \tag{1.27}$$

More general energy-momentum tensors for matter can be considered, e.g. the matter tensors of perfect fluid

$$T(M)^{ij} = (\rho + p)u^i u^j + pc^2 g^{ij} \tag{1.28}$$

where p denotes the pressure of matter. The conservation law of the whole energy-momentum and the equivalent equations of motion are also given by the equations (1.25) and (1.26).

The conservation law of the whole energy-momentum (1.25), the equations of motion (1.26), and the conservation law of mass (1.27) are given in co-variant form. The equations of motion (1.26) and the conservation of mass (1.27) can be rewritten in non-covariant form

$$\frac{1}{(-\eta)^{1/2}} \frac{\partial}{\partial x^k}\left((-\eta)^{1/2} T(M)^k{}_i\right) = \frac{1}{2}\frac{\partial g_{mn}}{\partial x^i} T(M)^{mn} \tag{1.29a}$$

$$\frac{1}{(-\eta)^{1/2}} \frac{\partial}{\partial x^k}\left((-\eta)^{1/2} \rho u^k\right) = 0 . \tag{1.29b}$$

The equations (1.29) give for a test particle, i.e. $p = 0$

$$\frac{d}{d\tau}\left(g_{ik} u^k\right) = \frac{1}{2}\frac{\partial g_{mn}}{\partial x^i} u^m u^n . \tag{1.30}$$

It follows by differentiation, the use of (1.11), and some elementary calculations

$$\frac{d^2 x^i}{d\tau^2} = -\Gamma(G)^i{}_{mn} \frac{dx^m}{d\tau}\frac{dx^n}{d\tau} \tag{1.31}$$

where $\Gamma(G)^i_{mn}$ denote the Christoffel symbols of g_{ij} .

It is worth mentioning that the equations for the gravitational field can be generalized including electro-magnetic fields, scalar fields, etc., by addition of the corresponding Lagrangians for these fields to (1.15) which will not be considered.

1.5 Field Equations Rewritten

It is sometimes useful for the applications of the gravitational theory to consider instead of g^{ij} symmetric tensors defined by

$$f^{ij} = \left(\frac{-G}{-\eta}\right)^{1/2} g^{ij} \tag{1.32a}$$

and

$$f_{ij} = \left(\frac{-G}{-\eta}\right)^{-1/2} g_{ij} \tag{1.32b}$$

yielding

$$f_{1k} f^{kj} = \delta_i^{\ j},\ f^{ik} f_{kj} = \delta^i_{\ j}\ . \tag{1.33}$$

Then, the equations for the gravitational field (1.23) can be rewritten

$$\left(f^{mn} f_{ik} f^{kj}{}_{/n}\right)_{/m} = 4\kappa T^j{}_i \tag{1.34}$$

where the energy-momentum tensor of gravitation has the form

$$T(G)^j{}_i = \frac{1}{8\kappa}\left[f_{mn} f_{kl} f^{jr}\left(f^{mk}{}_{/r} f^{nl}{}_{/i} - \frac{1}{2} f^{mn}{}_{/r} f^{kl}{}_{/i}\right) + \frac{1}{2}\delta^j{}_i L_G\right] \tag{1.35}$$

with

$$L_G = -f_{mn} f_{kl} f^{rs}\left(f^{mk}{}_{/r} f^{nl}{}_{/s} - \frac{1}{2} f^{mn}{}_{/r} f^{kl}{}_{/s}\right). \tag{1.36}$$

The energy-momentum tensor of perfect fluid is given by

$$T(M)^j{}_i = (\rho + p)\left(\frac{-F}{-\eta}\right)^{-1/2} f_{ik} u^j u^k + pc^2 \delta^j{}_i \tag{1.37}$$

where

$$F = \det\left(f_{ij}\right). \tag{1.38}$$

The relation (1.13) has the form

$$\left(\frac{-F}{-\eta}\right)^{-1/2} f_{mn}u^m u^n = -c^2 . \tag{1.39}$$

1.6 Field Strength and Field Equations

The equations of motion (1.31) of a test particle in the gravitational field are not generally co-variant.

A co-variant derivative of the four-vector $\left(u^i\right)$ of a test particle is

$$\frac{Du^i}{D\tau} = \frac{du^i}{d\tau} + \Gamma^i{}_{mn}u^m u^n . \tag{1.40}$$

$\Gamma^i{}_{mn}$ are the Christoffel symbols of the metric (1.1).

The equations of motion (1.31) can be rewritten by the substitution (1.40)

$$\frac{Du^i}{D\tau} = -\Delta\Gamma^i{}_{mn}u^m u^n \tag{1.41}$$

where

$$\Delta\Gamma^i{}_{mn} = \Gamma(G)^i{}_{mn} - \Gamma^i{}_{mn} . \tag{1.42}$$

Elementary calculations imply that $\Delta\Gamma^i{}_{jk}$ is a tensor of rank three. Hence, the equations of motion (1.41) for a test particle in the gravitational field $\left(g_{ij}\right)$ are generally co-variant. Similar to the equations of motion for a test particle in the electro-magnetic field where on the right hand side stands the Lorentz-force defined by the electro-magnetic field strength the tensor $\Delta\Gamma^i{}_{jk}$ in the equations (1.41) can be interpreted as gravitational field strength and the right hand side of (1.41) is the gravitational force.

Elementary calculations give

$$\frac{\partial g_{mn}}{\partial x^i} = \Gamma(G)^r{}_{mi}\, g_{nr} + \Gamma(G)^r{}_{ni}\, g_{mr} \, .$$

Hence, it follows

$$g_{mn/i} = \Delta\Gamma^r{}_{mi} g_{nr} + \Delta\Gamma^r{}_{ni} g_{mr} \, .$$

Therefore, we get

$$-g_{mk} g^{im}{}_{/j} = g^{im} g_{mk/j} = g^{im} g_{kn} \Delta\Gamma^n{}_{mj} + \Delta\Gamma^i{}_{kj} \, . \tag{1.43}$$

With the aid of (1.43) all the co-variant derivatives of g^{ij} can be replaced by the gravitational field strength. Elementary calculations give the Lagrangian

$$L_G = -2\left(\frac{-G}{-\eta}\right)^{1/2} g^{mn}\left(\Delta\Gamma^k{}_{lm}\Delta\Gamma^l{}_{kn} + g^{kl} g_{rs}\Delta\Gamma^r{}_{km}\Delta\Gamma^s{}_{ln} - \Delta\Gamma^r{}_{rm}\Delta\Gamma^s{}_{sn}\right) \tag{1.44}$$

The energy-momentum tensor of the gravitational field has the form

$$T(G)^j_i = \frac{1}{4\kappa}\left(\frac{-G}{-\eta}\right)^{1/2} g^{jn}\left(\Delta\Gamma^k{}_{ln}\Delta\Gamma^l{}_{ki} + g^{kl} g_{rs}\Delta\Gamma^r{}_{kn}\Delta\Gamma^s{}_{li} - \Delta\Gamma^r{}_{rn}\Delta\Gamma^s{}_{si}\right) + \frac{1}{16\kappa}\delta^j{}_i L_G \tag{1.45}$$

It follows for the equations of the gravitational field (1.23b)

$$\left[\left(\frac{-G}{-\eta}\right)^{1/2} g^{mn}\left(\Delta\Gamma^j{}_{in} + g^{jk} g_{il}\Delta\Gamma^l{}_{kn} - \delta^j{}_i\Delta\Gamma^k{}_{kn}\right)\right]_{/m} = -4\kappa T^j{}_i \, . \tag{1.46a}$$

The field equations (1.24) have the form

$$\left[\left(\frac{-G}{-\eta}\right)^{1/2} g^{mn}\left(\Delta\Gamma^j{}_{in} + g^{jk} g_{il}\Delta\Gamma^l{}_{kn}\right)\right]_{/m} = -4\kappa\left(T^j{}_i - \frac{1}{2}\delta^j{}_i T^m{}_m\right). \tag{1.46b}$$

Summarizing, we have written the theory of gravitation in flat space-time by the use of the field strength of gravitation similar to Maxwell's theory written with the aid of the electro-magnetic field strength.

1.7 Angular-Momentum

We will now derive the conservation law of the whole angular-momentum. Let us start from the conservation law of the whole energy-momentum (1.25b) which can be rewritten

$$\frac{1}{(-\eta)^{1/2}}\frac{\partial}{\partial x^m}\left((-\eta)^{1/2}\tilde{T}^{im}\right)+\Gamma^i{}_{mn}\tilde{T}^{mn}=0 \tag{1.47a}$$

where we have introduced the non-symmetric energy-momentum tensor

$$\tilde{T}^{ij}=\eta^{im}T^j{}_m\,. \tag{1.47b}$$

In an inertial frame, i.e. the metric tensor (η_{ij}) is constant and therefore $\Gamma^i{}_{jk}=0$ the relation (1.47a) implies a conservation law of the whole energy-momentum. Therefore, we get

$$P^i=\int(-\eta)^{1/2}\tilde{T}^{14}d^3x \quad \text{(i=1-4)} \tag{1.48}$$

Where P^i is a constant and the integration is taken over the whole space. Equation (1.47a) gives

$$\frac{1}{(-\eta)^{1/2}}\frac{\partial}{\partial x^m}\left(x^j(-\eta)^{1/2}\tilde{T}^{im}\right)=\tilde{T}^{ij}-x^j\Gamma^i{}_{mn}\tilde{T}^{mn}\,. \tag{1.49}$$

The field equations (1.23) imply

$$\tilde{T}^{ij}=\eta^{im}T^j{}_m=\frac{1}{4\kappa}\left[\left(\frac{-G}{-\eta}\right)^{1/2}g^{mn}\left(\eta^{ik}g_{kl}g^{jl}{}_{/n}-\frac{1}{2}\eta^{ij}g_{kl}g^{kl}{}_{/n}\right)\right]_{/m}.$$

The substitution of this relation into equation (1.49) and the subtraction from the arising from the same equation where i and j are exchanged yields

$$\begin{aligned}&\frac{1}{(-\eta)^{1/2}}\frac{\partial}{\partial x^m}\left(x^j(-\eta)^{1/2}\tilde{T}^{im}-x^i(-\eta)^{1/2}\tilde{T}^{jm}\right)\\&=A^{ijm}{}_{/m}-\left(x^j\Gamma^i{}_{mn}-x^i\Gamma^j{}_{mn}\right)\tilde{T}^{mn}\end{aligned} \tag{1.50}$$

with the contra-variant tensor

$$A^{ijk} = \frac{1}{4\kappa}\left(\frac{-G}{-\eta}\right)^{1/2} g^{km} g_{rs}\left(\eta^{is} g^{jr}{}_{/m} - \eta^{js} g^{ir}{}_{/m}\right). \tag{1.51}$$

It follows from equation (1.50) by the use of relations for the co-variant derivatives of tensors of order three

$$\begin{aligned} &\frac{1}{(-\eta)^{1/2}}\frac{\partial}{\partial x^m}\left((-\eta)^{1/2}\left(x^i \tilde{T}^{jm} - x^j \tilde{T}^{im} + A^{ijm}\right)\right) \\ &= \left(x^j \Gamma^i{}_{mn} - x^i \Gamma^j{}_{mn}\right)\tilde{T}^{mn} - \Gamma^i{}_{mn} A^{njm} - \Gamma^j{}_{mn} A^{inm} \end{aligned}. \tag{1.52}$$

These equations imply in uniformly moving frames the conservation law of the angular-momentum, i.e.

$$M^{ij} = \int (-\eta)^{1/2}\left[x^i \tilde{T}^{j4} - x^j \tilde{T}^{i4} + A^{ij4}\right] d^3x \ \left(i, j = 1,2,3,4\right) \tag{1.53}$$

is constant for all times. The first two expressions correspond to the usual definition of the angular momentum. To study the last expression we use the first part of the relation (1.43) and rewrite (1.51)

$$A^{ijk} = \frac{1}{4\kappa}\left(\frac{-G}{-\eta}\right)^{1/2} g_{mn} g^{kr} g^{sm}{}_{/r}\left(\eta^{in}\delta^j{}_s - \eta^{jn}\delta^i{}_s\right). \tag{1.54}$$

We now define the canonical momentum

$$\Pi_{ij} = \frac{1}{16\kappa}\frac{\partial L_G}{\partial g^{ij}{}_{/4}} \tag{1.55a}$$

implying

$$\Pi_{ij} = -\frac{1}{8\kappa}\left(\frac{-G}{-\eta}\right)^{1/2} g^{4k} g^{mn}{}_{/k}\left(g_{im} g_{jn} - \frac{1}{2} g_{ij} g_{mn}\right). \tag{1.55b}$$

The Hamiltonian is given by

$$H = \Pi_{mn} g^{mn}{}_{/4} - \frac{1}{16\kappa} L_G \,. \tag{1.56a}$$

Elementary computations give

$$H = -T(G)^4{}_4, \tag{1.56b}$$

i.e. H is the energy density of the gravitational field. It follows from (1.54) by the use of relation (1.55b)

$$A^{ij4} = 2\left(\delta^j{}_m \delta^i{}_n - \delta^i{}_m \delta^j{}_n\right)\eta^{mk} g^{nl} \Pi_{kl}. \tag{1.57}$$

We define for $i, j = 1,2,3,4$ the anti-symmetric four-matrices

$$\Sigma^{ij} = \left(\Sigma^{ij}{}_{mn}\right) = \left(\delta^j{}_m \delta^i{}_n - \delta^i{}_m \delta^j{}_n\right) \tag{1.58}$$

with the proper-values $0, \pm i$. The relation (1.57) can be rewritten

$$A^{ij4} = 2\Sigma^{ij}{}_{mn} \eta^{mk} g^{nl} \Pi_{lk}. \tag{1.59}$$

Hence, the last expression in equation (1.53) of the angular momentum can be interpreted as consequence of the spin of the gravitational field.

1.8 Equations of the Spin Angular Momentum

In this sub-chapter we follow along the lines of Papapetrou [Pap 51] who uses a method of Fock [Foc 39]. The following detailed calculations can be found in [Pet 91].

The equations of motion for matter (1.29a) can be written in the form:

$$\frac{1}{(-\eta)^{1/2}} \frac{\partial}{\partial x^m}\left((-\eta)^{1/2} T(M)^{im}\right) = -\Gamma(G)^i{}_{mn} T(M)^{mn} \tag{1.60}$$

where it is assumed that $T(M)^{ij}$ vanishes outside of a narrow tube which surrounds the world line of the test particle. The test particle describes a world line $X(t) = \left(X^i(t)\right)$ with $X^4(t) = ct$. Let us put in analogy to [Pap 51]

$$M^{ij} = u^4 \int (-\eta)^{1/2} T(M)^{ij} d^3x \tag{1.61a}$$

$$M^{kij} = -u^4 \int \left(x^k - X^k(t)\right)(-\eta)^{1/2} T(M)^{ij} d^3x \tag{1.61b}$$

$$\gamma^{ij} = -\frac{1}{u^4}\left(M^{ij4} - M^{ji4}\right). \tag{1.61c}$$

We obtain the equations of motion

$$\frac{d}{d\tau}\left(\frac{M^{i4}}{u^4}\right) = -\Gamma(G)^i{}_{mn} M^{mn} + \frac{\partial}{\partial x^k}\left(\Gamma(G)^i{}_{mn} M^{kmn}\right) \tag{1.62a}$$

and of the spin angular momentum

$$\begin{aligned}
&\frac{d}{d\tau}\gamma^{ij} + \frac{u^i}{u^4}\frac{d}{d\tau}\gamma^{j4} - \frac{u^j}{u^4}\frac{d}{d\tau}\gamma^{i4} \\
&= \left(\Gamma(G)^i{}_{mn} - \frac{u^i}{u^4}\Gamma(G)^4{}_{mn}\right) M^{jmn} \\
&+ \left(\Gamma(G)^j{}_{mn} - \frac{u^j}{u^4}\Gamma(G)^4{}_{mn}\right) M^{imn}
\end{aligned} \tag{1.62b}$$

Furthermore, we have

$$2M^{ijk} = -\left(\gamma^{ij}u^k + \gamma^{ik}u^j\right) + \frac{u^i}{u^4}\left(\gamma^{4j}u^k + \gamma^{4k}u^j\right) \tag{1.63a}$$

$$M^{ij4} + M^{ji4} = -u^i\gamma^{j4} - u^j\gamma^{i4} \tag{1.63b}$$

$$M^{i44} = -u^4\gamma^{i4} \tag{1.63c}$$

$$\begin{aligned}
M^{ij} &= \frac{u^i}{u^4}\left(\frac{u^j}{u^4}M^{44} - \frac{d}{d\tau}\left(\frac{M^{j44}}{u^4}\right) - \Gamma(G)^4{}_{mn} M^{jmn}\right) \\
&- \frac{d}{d\tau}\left(\frac{M^{ij4}}{u^4}\right) - \Gamma(G)^j{}_{mn} M^{imn}.
\end{aligned} \tag{1.63d}$$

Some of the relations (1.62) are identities. Therefore, we have eight equations (four equations (1.62a), three equations (1.62b) and one equation (1.13) for the eleven unknowns quantities M^{44}, u^i $(i=1,2,3,4)$, and γ^{ij} $(i,j=1,2,3)$. It is proved in [Pap 51] that γ^{ij} is the components of a tensor and the expression

$$m=-\frac{1}{c^3}\frac{1}{u^4}\left(M^{m4}+\Gamma(G)^m{}_{kl}\,\gamma^{k4}u^l\right)u_m \tag{1.64}$$

is a scalar where $u_i=g_{im}u^m$. We will now give a co-variant formulation of the equations (1.62).

In analogy to (1.40) we define the co-variant derivative

$$\frac{D}{D\tau}\gamma^{ij}=\gamma^{ij}{}_{/m}u^m=\frac{d}{d\tau}\gamma^{ij}+\Gamma^i{}_{mn}\gamma^{nj}u^m+\Gamma^j{}_{mn}\gamma^{in}u^m. \tag{1.65}$$

Let us introduce the anti-symmetric tensor

$$A^{ij}=\frac{D}{D\tau}\gamma^{ij}+\Delta\Gamma^i{}_{mn}\gamma^{mj}u^n+\Delta\Gamma^j{}_{mn}\gamma^{im}u^n. \tag{1.66}$$

Then, we have by (1.62b), (1.63a), (1.42) and (1.65)

$$A^{ij}+\frac{u^i}{u^4}A^{j4}-\frac{u^j}{u^4}A^{i4}=0. \tag{1.67}$$

When we multiply (1.67) with u_j we get

$$\frac{1}{u^4}A^{i4}=-\frac{u^m}{c^2}\left(\frac{u^i}{u^4}A^{m4}+A^{im}\right). \tag{1.68}$$

By the use of the last two relations we get the co-variant form of (1.62b)

$$A^{ij}+\frac{1}{c^2}u_m\left(u^jA^{im}-u^iA^{jm}\right)=0. \tag{1.69}$$

We will now give (1.62a) in co-variant form and write (1.63d) for $j=4$ with the aid of (1.63a), (1.63c), $M^{4ij}=0$, (1.65) and (1.66)

$$M^{i4}+\Gamma(G)^i{}_{mn}\,\gamma^{m4}u^n=A^{i4}+\frac{u^i}{u^4}\left(M^{44}+\Gamma(G)^4{}_{mn}\,\gamma^{m4}u^n\right). \tag{1.70}$$

We get by multiplying this relation with $\frac{u^i}{u^4}$ and the use of (1.64)

$$\frac{1}{\left(u^4\right)^2}\left(M^{44} + \Gamma(G)^4{}_{mn}\,\gamma^{m4}u^n\right) = mc + \frac{1}{c^2}\frac{u_m}{u^4}A^{m4}\,. \tag{1.71}$$

Hence, we get from (1.70) by the use of (1.71) and (1.68)

$$\frac{1}{u^4}M^{i4} = mcu^i - \frac{1}{u^4}\Gamma(G)^i{}_{kl}\,\gamma^{k4}u^l - \frac{1}{c^2}u_k A^{ik}\,.$$

Now, it follows from (1.62a) by the use of (1.68), (1.61) and elementary calculations

$$\begin{aligned}&\frac{d}{d\tau}\left(mcu^i - \frac{1}{c^2}u_k A^{ik}\right) + \Gamma(G)^i{}_{kl}\,u^k\left(mcu^l - \frac{1}{c^2}u_r A^{lr}\right)\\ &+\left(\frac{\partial}{\partial x^m}\Gamma(G)^i{}_{lk} + \Gamma(G)^i{}_{nm}\,\Gamma(G)^n{}_{kl}\right)\gamma^{mk}u^l\\ &= 0\end{aligned} \tag{1.72}$$

The introduction of the co-variant derivative of a four-vector gives

$$\begin{aligned}&\frac{D}{D\tau}\left(mcu^i - \frac{1}{c^2}u_m A^{im}\right) + \Delta\Gamma^i{}_{mn}u^m\left(mcu^n - \frac{1}{c^2}u_k A^{nk}\right)\\ &+\frac{1}{2}R^i{}_{mnk}\gamma^{nm}u^k = 0\end{aligned} \tag{1.73}$$

where $R^i{}_{mnk}$ is the curvature tensor of g_{ij}. Although the equations (1.62a) and (1.62b) are identical with those of general relativity the co-variant forms (1.73), (1.69) together with (1.66) are different from those of general relativity [Pap 51]. γ^{ij} which is defined by (1.61c) is not the spin in flat space-time theory of gravitation. The spin of a particle must be defined by

$$\begin{aligned}S^{ij} = &\int\left(x^i - X^i(t)\right)(-\eta)^{1/2}\,T\left(\tilde{M}\right)^{j4} d^3x\\ &-\int\left(x^j - X^j(t)\right)(-\eta)^{1/2}\,T\left(\tilde{M}\right)^{i4} d^3x\end{aligned}\,. \tag{1.74}$$

In Einstein's theory the motion of a spin in free fall can be described according to the equations of parallel transport (see e.g. [Wei 72]. This is not possible by the use of flat space-time theory of gravitation.

1.9 Transformation to Co-Moving Frame

In the previous sub-chapter we have seen that there are not enough equations for the spin components. Schiff [Sch 80] remarked that one has to transform the equations of spin components to the co-moving frame, i.e. to the frame of the gyroscope. We use the considerations of Petry [Pet 86] to transform from a preferred frame Σ' with $(\eta_{ij}')=diag(1,1,1,-1)$ to a non-preferred frame Σ moving with velocity $v'=(v^1{}',v^2{}',v^3{}')$ relative to the frame Σ' . Let $\left(X^1{}'(t'),X^2{}'(t'),X^3{}'(t')\right)$ be the distance vector of Σ from Σ'. Then,

$$\frac{d}{dt'}X^i{}'(t')=-v^i{}'. \tag{1.75}$$

The transformations of quantities in Σ' to the corresponding ones in the co-moving frame Σ are given in [Pet 86]

$$x^i{}'=x^i+\left(\gamma^{-1}-1\right)\frac{\left(x,\frac{v'}{c}\right)}{\left|\frac{v'}{c}\right|^2}\frac{v^i{}'}{c}+X^i{}'(t'),\ \ dt'=\gamma dt \tag{1.76a}$$

with

$$\gamma=\left(1-\left|\frac{v'}{c}\right|^2\right)^{-1/2}. \tag{1.76b}$$

It is sufficient to consider (1.76) up to quadratic expressions in the absolute value of the velocity $|v'|$, i.e.

$$x^i{}'\approx x^i-\frac{1}{2}\left(x,\frac{v'}{c}\right)\frac{v^i}{c}+X^i{}',\ dt'\approx\left(1+\frac{1}{2}\left|\frac{v'}{c}\right|^2\right)dt. \tag{1.77}$$

In the frame Σ' we consider equation (1.62b), multiplied with $d\tau/dt'$, the use of (1.63a) and $u^i{}'/u^4{}'=v^i{}'/c$, i.e.

$$\begin{aligned}&\frac{d}{dt'}\gamma^{ij}{}'+\frac{v^i{}'}{c}\frac{d}{dt'}\gamma^{j4}{}'-\frac{v^j{}'}{c}\frac{d}{dt'}\gamma^{i4}{}'-\left(\Gamma(G)^i{}'_{mn}-\frac{v^i{}'}{c}\Gamma(G)^4{}'_{mn}\right)\gamma^{jm'}v^n{}'\\&+\left(\Gamma(G)^j{}'_{mn}-\frac{v^j{}'}{c}\Gamma(G)^4{}'_{mn}\right)\gamma^{im}{}'v^n{}'\\&+\left(\frac{v^j{}'}{c}\Gamma(G)^i{}'_{mn}-\frac{v^i{}'}{c}\Gamma(G)^j{}'_{mn}\right)\gamma^{4m}{}'v^n{}'=0\end{aligned} \tag{1.78}$$

Furthermore, it follows

$$\gamma^{ij}{}'=\gamma^{mn}\frac{\partial x^i{}'}{\partial x^m}\frac{\partial x^j{}'}{\partial x^n}.$$

We get after some calculations for the spin tensor γ^{ij} in Σ

$$\gamma^{ij}{}'\approx\gamma^{ij}+\frac{v^i{}'}{c}\gamma^{4j}+\frac{v^j{}'}{c}\gamma^{i4}-\frac{1}{2}\left(\frac{v^i{}'}{c}\sum_{k=1}^{3}\gamma^{kj}\frac{v^k{}'}{c}+\frac{v^j{}'}{c}\sum_{k=1}^{3}\gamma^{ik}\frac{v^k{}'}{c}\right) \tag{1.79a}$$

$$\gamma^{i4}{}'\approx\left(1+\frac{1}{2}\left|\frac{v'}{c}\right|^2\right)\gamma^{i4}-\frac{1}{2}\frac{v^i{}'}{c}\sum_{k=1}^{3}\gamma^{k4}\frac{v^k{}'}{c}. \tag{1.79b}$$

If we substitute (1.79) into (1.78) and neglect expressions of the form $\Gamma^{\bullet}{}_{\bullet\bullet}\dfrac{v^{\bullet}}{c}\dfrac{v^{\bullet}}{c}$ it follows by elementary calculations

$$\frac{d}{dt'}\gamma^{ij}=\Gamma(G)^4{}_{44}\left(-v^i{}'\gamma^{j4}+v^j{}'\gamma^{i4}\right)-\left(\sum_{k=1}^{3}\Omega^{ik}\gamma^{jk}-\Omega^{jk}\gamma^{ik}\right) \tag{1.80a}$$

where

$$\Omega^{ij}=-\left(\sum_{k=1}^{3}\Gamma(G)^i{}_{jk}v^k{}'+\Gamma(G)^i{}_{j4}c-\Gamma(G)^4{}_{j4}v^i{}'+\frac{1}{2}\Gamma(G)^i{}_{44}v^j{}'+\frac{1}{2}\Gamma(G)^j{}_{44}v^i{}'\right) \tag{1.80b}$$

We will now apply the result to the spin angular momentum of a test particle in the gravitational field of a spherically symmetric body in the preferred frame

Σ' with mass M and angular velocity ω. It holds in Σ' up to linear approximations

$$\begin{aligned} g_{ij}' &= \delta_{ij}\left(1+2\frac{kM}{c^2 r}\right),(i,j=1,2,3) \\ &= -\left(1-2\frac{kM}{c^2 r}\right),(i=j=4) \\ &= -\frac{2kJ}{c^3}\frac{1}{r^3}[\omega\times x'],(i=4,j=1,2,3;i=1,2,3,j=4) \end{aligned} \tag{1.81}$$

where J is the momentum of inertia. We get by elementary computations

$$r=\left(\sum_{k=1}^{3}\left|x^k\right|^2\right)^{1/2},\Omega^{11}=\Omega^{22}=\Omega^{33}\approx\frac{kM}{c^2}\frac{1}{r^3}(x',v'),\Omega^{ij}=-\Omega^{ji},i\neq j.$$

Put

$$\Omega=\left(\Omega^{23},\Omega^{31},\Omega^{12}\right)$$

then, we have

$$\Omega\approx-\frac{3}{2}\frac{kM}{c^2}\frac{1}{r^3}[x'\times v']+\frac{kJ}{c^2}\frac{1}{r^3}\left(\omega-3\frac{(x',\omega)}{r^2}x'\right). \tag{1.82}$$

We define $\gamma=\left(\gamma^{23},\gamma^{31},\gamma^{12}\right)$. Relation (1.81a) is rewritten in the form

$$\frac{d}{dt'}\gamma\approx 2\frac{kM}{c^2 r^3}(x',v')\gamma-[\Omega\times\gamma]. \tag{1.83a}$$

By the use of the law of Newton

$$\frac{dv'}{dt'}\approx -kM\frac{x'}{r^3}$$

we get

$$\frac{d}{dt'}\gamma\approx-2\left(\frac{1}{c}\frac{dv'}{dt'},\frac{v'}{c}\right)\gamma-[\Omega\times\gamma]$$

$$\Omega \approx \frac{3}{2}\left[\frac{1}{c}\frac{dv'}{dt'}\times\frac{v'}{c}\right]+\frac{kJ}{c^2r^3}\left(\omega-3\frac{(x',\omega)}{r^2}x'\right). \tag{1.83b}$$

We consider instead of γ the spin. We get in Σ by the use of the standard transformation formula, considering only expressions which are quadratic in the velocity and linear in the expression $\frac{kM}{c^2r}$, the use of (1.81)

$$\begin{aligned} g_{ij} &\approx \delta_{ij}\left(1+2\frac{kM}{c^2r}\right)-\frac{v^{i\,'}}{c}\frac{v^{j\,'}}{c},\ (i,j=1,2,3)\\ &\approx \frac{v^{i\,'}}{c},\ (i=1,2,3;\ j=4)\\ &\approx \frac{v^{j\,'}}{c},\ (i=4;\ j=1,2,3)\\ &\approx -\left(1-2\frac{kM}{c^2r}\right)(i=j=4). \end{aligned} \tag{1.84}$$

The metric tensor has the form

$$\begin{aligned} \mathrm{\eta_{ij}} &= \mathrm{\delta_{ij}},\ \ (\mathrm{i,j}=1,2,3,4)\\ &= \frac{\mathrm{v^{i\,'}}}{\mathrm{c}},\ \ (\mathrm{i}=1,2,3;\ \mathrm{j}=4)\\ &= \frac{\mathrm{v^{j\,'}}}{\mathrm{c}},\ \ (\mathrm{i}=4;\ \mathrm{j}=1,2,3)\\ &= -\left(1-\left|\frac{\mathrm{v'}}{\mathrm{c}}\right|^2\right),\ \ (\mathrm{i}=\mathrm{j}=4). \end{aligned} \tag{1.85}$$

We get from the definition (1.74), (1.85) and (1.61) for i,j=1,2,3

$$\begin{aligned} S^{ij} &= \eta^{jm}g_{mk}\int x^i(-\eta)^{1/2}T(M)^{k4}\,d^3x-\eta^{im}g_{mk}\int x^j(-\eta)^{1/2}T(M)^{k4}\,d^3x\\ &\approx\left(1+2\frac{kM}{c^2r}\right)\gamma^{ij}. \end{aligned}$$

Hence, it holds

$$\gamma \approx \left(1 - 2\frac{kM}{c^2 r}\right) S \,. \tag{1.86}$$

We have by the substitution of (1.86) into the relation (1.83a)

$$\frac{d}{dt'}\left(\left(1 - 2\frac{kM}{c^2 r} + \left|\frac{v'}{c}\right|^2\right) S\right) \approx \left(1 - 2\frac{kM}{c^2 r}\right)\left[\Omega \times S\right].$$

By the use of the conservation law of energy

$$\frac{1}{2}\left|\frac{v'}{c}\right|^2 - \frac{kM}{c^2 r} = const$$

we get

$$\frac{d}{dt} S \approx -\left[\Omega \times S\right]. \tag{1.87}$$

Equation (1.87) gives the precession of the spin of a test particle with constant angular velocity. It agrees with the corresponding result of general relativity [Sch 60]. The angular momentum of a gyroscope processes without changing in magnitude. The results about the spin angular momentum and the gyroscope agree with those of general relativity.

All these results of the sub-chapters 1.8 and 1.9 can be found in [Pet 91]. For experimental technical problems compare Will [Wil 81].

The results of chapter I about the theory of gravitation in flat space-time can be found in the articles of Petry [Pet 79, 81a, 82,93b].

It is worth to mention the article [Cah 07] of Cahill who has studied a theory of gravitation with application to cosmology by a method which is totally different from general relativity and any bi-metric theory.

1.10 Approximate Solution in Empty Space

By the use of general relativity approximate solutions in empty space are received by linearization of the non-linear equations. This can also be considered by the use of flat space-time theory of gravitation as will be seen in sub-chapter 2.2. Therefore, we will study the linearization of the gravitational field. We start from the gravitational theory in flat space-time (1.23) together with the conservation of the whole energy-momentum (1.25). Formula (1.23b) of the field equations implies by the use of covariant differentiation, the conservation law (1.25a) and the use of the pseudo-Euclidean geometry (1.5)

$$\frac{\partial}{\partial x^j} D_i^j - \frac{1}{2}\frac{\partial}{\partial x^i} D_m^m = 0 \quad \text{(i=1-4).} \tag{1.88}$$

Relation (1.88) gives by the use of linearization, i.e.

$$g^{ij} = \eta^{ij} + \Delta g^{ij}$$

the linearized expression

$$D_i^j = \eta^{mn}\eta_{ik}\frac{\partial}{\partial x^m}\frac{\partial}{\partial x^n}\Delta g^{kj}.$$

Therefore, relation (1.88) can be written in the form

$$\eta^{mn}\frac{\partial}{\partial x^m}\frac{\partial}{\partial x^n}\left\{\eta_{ik}\frac{\partial}{\partial x^j}\Delta g^{kj} - \frac{1}{2}\frac{\partial}{\partial x^i}\left(\eta_{kl}\Delta g^{kl}\right)\right\} = 0.$$

The operator in front of the bracket is the wave operator. Hence we get

$$\eta_{ik}\frac{\partial}{\partial x^j}\Delta g^{kj} - \frac{1}{2}\frac{\partial}{\partial x^i}\left(\eta_{kl}\Delta g^{kl}\right) \text{ (i=1-4).} \tag{1.89}$$

Relation (1.89) is identical with the result of general relativity (see e.g.[Rob 68], p. 256, [Sex 83], $p.$175) which is used for many applications.The derivation of relation (1.89) in empty space (no matter) uses the fact that in empty space a gravitational field exists which must be considered. The quite different study of linear approximations of the gravitational field by flat space-time theory of gravitation and general relativity follows from the different sources in the theories. Flat space-time theory of gravitation has the whole energy-momentum as source whereas general relativity has only the matter tensor. In general relativity the energy-momentum is not a tensor which implies many difficulties (see the extensive study of Logunov and co-workers (see e.g. [Log 86], [Den 82,84]).

A comparison of the theory of gravitation in flat space-time and the theory of general relativity is given in [Pet 14a].